ELEMENTARY
ANALYTICAL CONICS

BY

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PREFACE

THE object of this book is to frame an elementary course in Analytical Conics primarily for the use of schools where advanced course candidates are being prepared for Higher Certificate Examinations and for scholarships awarded by the State and Local Education Authorities throughout the country. It should also provide a sound foundation for the work of students who intend to compete for mathematical scholarships at the universities.

Especial emphasis has throughout been laid on the use of parametric coordinates. Many worked examples have also been given to familiarize the student with the processes usually employed in solving problems set at examinations.

The author desires to acknowledge his indebtedness to C. H. Sampson, Esq., Principal of Brasenose College, Oxford, and to Dr. F. S. Macaulay, whose work in the pure geometry of conics is so well known amongst mathematicians, for having first aroused his interest in this subject by their teaching. He has also to express his gratitude to his colleague Mr. J. E. Tarver, B.A., for valuable advice, and assistance in revising proofs and collecting examples. To Mr. R. K. Melluish, of Rossall School, and to Mr. F. J. Bollands, who prepared the diagrams for the book, he owes a similar acknowledgment for help given.

Finally he has to thank the undermentioned examining boards for permission to use examples from their Higher Certificate papers. These appear throughout the book with the letter in brackets appended to them. University of Bristol [B.], Cambridge Local Syndicate [C.], University of Durham [D.], Joint Matriculation Board [J.], University of London [L.], Delegates of the Oxford Local Examinations [O.], Oxford and Cambridge Schools Examination Board [O. and C.], Central Welsh Board [W.].

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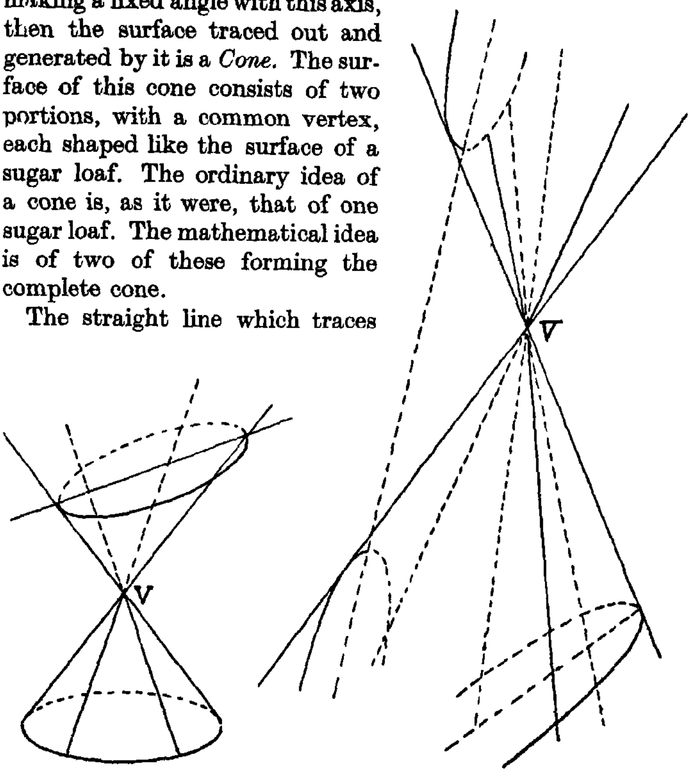
ERRATA

p. 27, Ex. 4: read $\tan\alpha = \frac{3}{2}$, $\tan\beta = \frac{2}{3}$.

p. 83, Ex. 7, 1 (5): read $3x^2 - 7xy + 2y^2 = 1$.

INTRODUCTION

If a straight line of unlimited length has one point on it fixed and rotates round a fixed axis through this point, always making a fixed angle with this axis, then the surface traced out and generated by it is a *Cone*. The surface of this cone consists of two portions, with a common vertex, each shaped like the surface of a sugar loaf. The ordinary idea of a cone is, as it were, that of one sugar loaf. The mathematical idea is of two of these forming the complete cone.



The straight line which traces out a conical surface is called the *Generating Line*. Any particular position of this line is called a *Generator* of the cone. Every generator passes through the *Vertex* V of the cone.

Conic Sections are the curves formed when a plane cuts the surface of a complete cone in every possible way. As the curve of intersection necessarily lies on the intersecting plane, it follows that every conic section must be a plane figure.

The nature and shape of the conic sections will vary. The two figures on the previous page show six cases which may arise.

I. Where the intersecting plane passes through the vertex V of the cone and has a greater slope with its axis than any generator.

The section in this case is a *Point*, i e. the vertex itself.

II. Where the intersecting plane passes through the vertex V of the cone and contains a generator.

In this case, generally, the intersecting plane cuts the surface of the cone again in another generator. The section would thus be *Two Straight Lines*.

III. Where the intersecting plane does not go through the vertex but is perpendicular to the axis of the cone.

In this case the intersecting plane cuts one branch only of the cone and the section will be a *Circle*.

IV. Where the intersecting plane is parallel to a generator.

In this case again only one branch of the cone is cut. The curve of intersection between the conical surface and the plane extends to infinity in two directions and is known as a *Parabola*.

V. Where the intersecting plane has a greater slope with the axis of the cone than that of any generator but does not pass through the vertex.

In this case also only one branch of the cone is cut and, unless the intersecting plane is perpendicular to the axis as in III, the section is called an *Ellipse*.

The orbits of the planets, including the earth, round the sun are all elliptical.

VI. Where the intersecting plane has a lesser slope with the axis of the cone than that of any generator but does not pass through the vertex.

In this case both branches of the cone are cut and two similar curves are formed, one on each branch, each of them going to infinity in two directions. Such a section is called a *Hyperbola*.

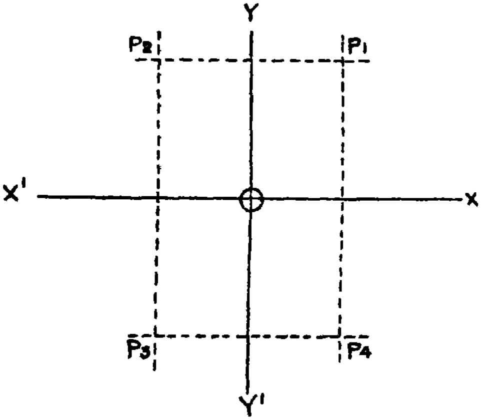
The properties of each of these six conic sections will now be investigated in turn.

I

THE POINT

1. Coordinates of a point

TAKE a point O and through it draw two straight lines XOX', YOY' at right angles to each other. In practice these lines are drawn parallel to the edges of the paper used. They are called *Axes of Coordinates*, or simply the Axes, and their point of intersection O is known as the *Origin*.



Take any point P in the plane of the paper and suppose that its perpendicular distance from the axis YOY' is a known length of d units.

Then, as the figure shows, the point P could be on either of the two dotted lines drawn parallel to the axis YOY' and each at a distance d away from it.

If the quantity d is positive, we assign the point P to the dotted line which is to the right of YOY' and parallel to it; but if d is negative, then the point is assigned to the dotted line to the left of YOY' also parallel to it.

In a similar way, if P is known to be perpendicularly distant from the axis XOX' a known length of e units, then P must lie on one of the two dotted lines parallel to the axis XOX' which are each a distance e away from it.

If the quantity e is positive we assign the point to the dotted line parallel to XOX' and above it; if e is negative it is assigned to the dotted line parallel to XOX' which lies below it.

The line XOX', usually drawn horizontal, is sometimes called the x-axis or axis of x. The line YOY', which is vertical, is the y-axis or axis of y.

If we just know that a point P is at a distance d from the y-axis and e from the x-axis, then it might have any one of the four positions P_1, P_2, P_3, P_4 in the figure. But, if the signs of d and e are both known, then it can be determined at once which of these four positions is the one required.

$d +$ and $e +$ signifies that the point is d to the right of the y-axis and e above the x-axis, i.e. its position is P_1.

$d -$ and $e +$ signifies that the point is d to the left of the y-axis and e above the x-axis, i.e. its position is P_2.

$d -$ and $e -$ signifies that the point is d to the left of the y-axis and e below the y-axis, i.e. its position is P_3.

$d +$ and $e -$ signifies that the point is d to the right of the y-axis and e below the y-axis, i.e. its position is P_4.

The four divisions of area made by the axes are generally called *Quadrants*. Reckoning in a direction opposite to that of the hands of a watch, XOY contains the first quadrant, YOX' the second, $X'OY'$ the third, and $Y'OX$ the fourth quadrant.

Conversely, for any point in the first quadrant the distances from the axes are both positive, while for any point in the third quadrant they are both negative. In the second quadrant the distance from the y-axis is negative and from the x-axis is positive. In the fourth quadrant the reverse of this is the case, i.e. the distance from the y-axis is positive and that from the x-axis is negative.

This allocation of signs is purely conventional, but it has to be thoroughly understood from the outset.

The perpendicular distances of any point from the axes of *coordinates, due regard being paid to sign conventions, are called its coordinates.*

The horizontal distance of a point from the y-axis, supposed vertical, with the proper sign attached, is called its x-coordinate or its *abscissa.*

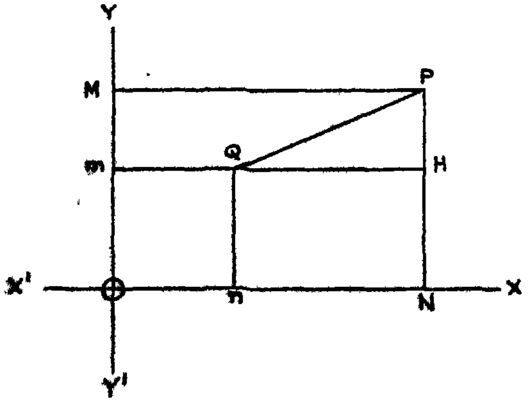
Similarly, the vertical distance of a point from the x-axis, supposed horizontal, with the proper sign attached, is called its y-coordinate or simply its *ordinate.*

From what has now been said it is clear that when we know the abscissa and ordinate of any point, that point can at once be determined and plotted graphically if desired. The coordinates of a point are usually written within brackets. Thus $(a, -b)$ indicates a point whose abscissa is $+a$ and ordinate $-b$. It obviously lies somewhere in the fourth quadrant.

The student who has not had previous experience of dealing with graphs in algebra should now take a piece of graph paper, draw two lines at right angles in the middle of it, and then, taking a scale of $\frac{1}{10}$ in. or 1 cm. to a unit, plot several points such as $(2, -4)$, $(-\frac{1}{2}, -\frac{3}{4})$, $(-5, 2\frac{1}{4})$, $(1, -\frac{2}{3})$, etc.

2. The distance between two points

P and Q are any two points (x_1, y_1), (x_2, y_2), the distance between which it is required to find.



Draw PM, Qm parallel to the x-axis and PN, Qn parallel to the y-axis.

Produce mQ to meet PN in H.

Then $PQ^2 = PH^2+HQ^2$ by Pythagoras.

But $PH = PN-HN = PN-Qn = y_1-y_2$

and $HQ = Hm-Qm = ON-On = x_1-x_2$.

$$\therefore\ \mathbf{PQ^2 = (x_1-x_2)^2+(y_1-y_2)^2}.$$

The practical rule for getting the square of the distance between two points whose coordinates are given is

(1) Subtract the abscissae and square.
(2) Subtract the ordinates and square.
(3) Add the results of (1) and (2).

If Q had been the origin O, the point (0, 0), it is clear that $OP^2 = x_1^2+y_1^2$.

Hence the square of the distance of any point from the origin is the sum of the squares of its coordinates.

Typical Examples

Ex. 1. Find the distance between the points $(-3, 5)$, $(2, -7)$.

$(-3-2)^2 = 5^2 = 25$, $(5+7)^2 = 12^2 = 144$. $25+144 = 169$.

Hence the distance is $\sqrt{169} = 13$.

Ex. 2. Show that the points $(1\frac{1}{2}, 6)$, $(-6, -1\frac{1}{2})$, $(3, 1\frac{1}{2})$ are the vertices of a right-angled triangle.

$$(1\tfrac{1}{2}+6)^2+(6+1\tfrac{1}{2})^2 = 2(7\tfrac{1}{2})^2 = \tfrac{225}{2},$$

$$(-6-3)^2+(-1\tfrac{1}{2}-1\tfrac{1}{2})^2 = 9^2+3^2 = 90,$$

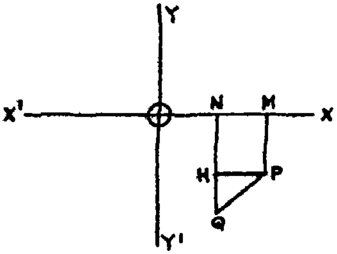
$$(3-1\tfrac{1}{2})^2+(1\tfrac{1}{2}-6)^2 = \tfrac{9}{4}+\tfrac{81}{4} = \tfrac{45}{2}.$$

But $90+\frac{45}{2} = \frac{225}{2}$.

$\therefore$ by the converse of Pythagoras, the triangle formed by joining the three points is right-angled.

Ex. 3. Find the lengths of the projections on the axes of the line joining the two points $(a, -b)$, $(b, -a)$.

Let P be the point $(a, -b)$, Q the point $(b, -a)$.



Draw PM, QN perpendicular to the x-axis and PH perpendicular to QN.

Then the projection of PQ on the x-axis is MN and its projection on the y-axis $= QH$.

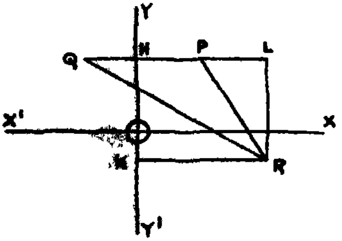
Now

$$a = OM = ON + NM = b + NM,$$

$$\therefore \; NM = a - b.$$

Also $\quad QH = QN - HN = -a - PM = -a + b.$

Ex. 4. What is the area of the triangle formed by joining the three points $(2, 3)$, $(-1\frac{1}{2}, 3)$, $(4, -1)$?



Plot the three points P, Q, R as in the figure.

Let PQ cut the y-axis in H.

Draw RK perpendicular to the y-axis.

Also draw RL perpendicular to QP produced.

Since P and Q are each a distance 3 from OX,

$\therefore$ PQ is parallel to the x-axis.

$$\begin{aligned}\text{Hence area of } \Delta RPQ &= \tfrac{1}{2} PQ \,.\, RL \\ &= \tfrac{1}{2}(PH + QH)KH \\ &= \tfrac{1}{2}(PH + QH)(OK + OH) \\ &= \tfrac{1}{2}(2 + 1\tfrac{1}{2})(1 + 3) \\ &= 7.\end{aligned}$$

EXAMPLES 1

1. Plot the points $(2, 4)$, $(-6, 1)$, $(3, 2)$, $(-1, -2)$, $(3, 6)$, $(-2\frac{1}{2}, -3\frac{1}{2})$, $(3, 0)$, $(0, -2)$.

2. In the preceding question produce the line joining the points $(2, 4)$, $(3, 2)$ to cut the axes of coordinates, and find by calculation

and by measurement the distances from the origin of the two points of intersection.

3. What can be inferred about the following pairs of points without plotting them? (i) $(2, 5)$, $(-2, -5)$. (ii) $(3, 4)$, $(4, 3)$. (iii) $(3, 1)$, $(3, -1)$.

4. Find the distance between the following pairs of points: (i) $(0, 3)$ and $(-2, 1)$. (ii) $(-3, -3)$ and $(4, -4)$. (iii) $(1\frac{1}{4}, -1\frac{1}{2})$ and $(-2\frac{3}{4}, -3\frac{1}{2})$. (iv) $(a+b, a-b)$ and $(a+2b, a-2b)$.

5. The coordinates of two points are $(3, -1)$, $(-2, 3)$, and the abscissa of a point on the line joining them is 1. What is the ordinate of this point?

6. If the distance between the two points $(x, 4)$, $(3, 2)$ is 2, find x.

7. Without finding the actual distances, give a reason why the distance between the two points $(6, 2)$, $(-1, 3)$ should be the same as that between the two points $(5, 5)$, $(-2, 4)$.

8. If $x_1^2+y_1^2 = x_2^2+y_2^2$, what geometrical fact can be inferred about the two points (x_1, y_1), (x_2, y_2)?

9. If the triangle whose vertices are the points $(1, 2)$, $(1, 3)$, $(1+x, 2\frac{1}{2})$ is equilateral, find x.

10. What is the area of the triangle formed by joining the three points $(4, 1)$, $(0, -2)$, $(4, -1)$?

11. The points $(3, -1)$, $(3, -2)$, $(5, 4)$, and (x, y) form a parallelogram. What must x and y be?

12. Find the distance between the following pairs of points:

(1) $(a\cos\theta, a\sin\theta)$, $(a\cos\phi, a\sin\phi)$.

(2) $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$.

(3) $(a\cos\theta, b\sin\theta)$, $(a\cos\phi, b\sin\phi)$.

(4) $\left(at_1, \frac{a}{t_1}\right)$, $\left(at_2, \frac{a}{t_2}\right)$.

13. Find the area of the triangle whose vertices are the points $(0, 0)$, $(at_1^2, 2at_1)$, $(at_1^2, -2at_1)$.

14. Find the coordinates of the circumcentre of the triangle whose vertices are the points $(2, 3)$, $(1, -4)$, $(-6, -2)$.

15. Show that the points with coordinates $(8, 8)$, $(13, -4)$, $(1, -9)$, $(-4, 3)$ are at the corners of a square. Find the area of the part of this square which lies in the positive quadrant. [J.]

3. The coordinates of certain points

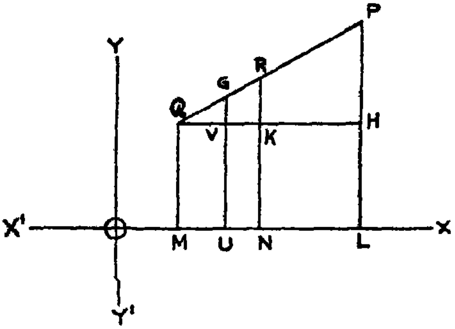
To find the coordinates of the middle point of the line joining two fixed points (x_1, y_1), (x_2, y_2).

In the figure P and Q are the points (x_1, y_1), (x_2, y_2).

Let R be the middle point of PQ and its coordinates x and y.

Draw PL, QM, RN perpendicular to the x-axis.

Draw QKH parallel to the x-axis cutting RN in K and PL in H.



Since RK is parallel to PH and R is the middle point of PQ,

$$\therefore\ RK = \tfrac{1}{2}PH.$$

$$\therefore\ y = RN = RK+KN = \tfrac{1}{2}PH+QM = \tfrac{1}{2}(PL-HL)+QM$$
$$= \tfrac{1}{2}(PL-QM)+QM = \tfrac{1}{2}(PL+QM) = \tfrac{1}{2}(y_1+y_2).$$

Similarly, by drawing perpendiculars from P, Q, and R on the y-axis, it can be proved that $x = \tfrac{1}{2}(x_1+x_2)$.

Thus the coordinates of the middle point of the line joining any two points are the arithmetic means of the abscissae and ordinates of these points.

To find the coordinates of a point which divides the line joining two given points in the ratio 2 : 1.

Let the two given points P, Q be (x_1, y_1), (x_2, y_2) and, with the same figure as before, let $PG = 2GQ$.

Draw GVU perpendicular to the x-axis and cutting QKH in V.

Then from parallels $GV : PH = GQ : PQ = 1 : 3$.

Let G be the point (x, y).

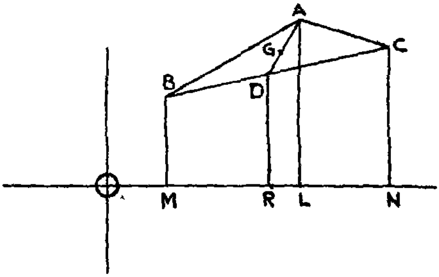
Then

$$y = GU = GV + VU = \tfrac{1}{3}PH + QM = \tfrac{1}{3}(PL - HL) + QM$$
$$= \tfrac{1}{3}(PL - QM) + QM = \tfrac{1}{3}PL + \tfrac{2}{3}QM$$
$$= \frac{2y_2 + y_1}{3}.$$

Similarly, by drawing perpendiculars from P, Q, and G to the y-axis we should get

$$x = \frac{2x_2 + x_1}{3}.$$

To find the coordinates of the centroid of a triangle.



Let ABC be a triangle and its vertices A, B, C the points (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively.

Let D be the middle point of BC. Join AD and divide it at G so that $AG = 2GD$. Then G is the centroid of the triangle ABC.

Suppose G is the point (x, y).

Draw perpendiculars AL, BM, CN, DR to the x-axis.

Then, D being the middle point of BC, $DR = \tfrac{1}{2}(y_2 + y_3)$.

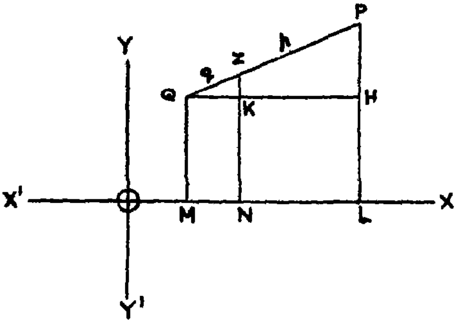
And, as above, the ordinate of $G = \dfrac{AL + 2DR}{3}$.

$$\therefore\ y = \frac{y_1 + y_2 + y_3}{3}.$$

In the same way it may be proved that $x = \dfrac{x_1+x_2+x_3}{3}$.

Thus the abscissa and ordinate of the centroid of a triangle are the arithmetic means respectively of the abscissae and ordinates of its vertices.

To find the coordinates of a point which divides the line joining two fixed points (x_1, y_1), (x_2, y_2) in a given ratio $p : q$.



Let P, Q be the two fixed points (x_1, y_1), (x_2, y_2) and Z the point (x, y) which divides the line PQ in the given ratio $p : q$.

Draw perpendiculars PL, QM, ZN to the x-axis.

Draw QKH parallel to the x-axis and cutting PL, ZN in H and K respectively.

Then from parallels $ZK : PH = q : p+q$.

$$\therefore\ (p+q)ZK = qPH$$

$$\therefore\ (p+q)(ZN-KN) = q(PL-HL)$$

$$\therefore\ (p+q)ZN = (p+q)QM+qPL-qQM$$

$$= pQM+qPL.$$

$$\therefore\ y = ZN = \frac{py_2+qy_1}{p+q}.$$

And, similarly, it may be proved that

$$x = \frac{px_2+qx_1}{p+q}.$$

If the division were external, i.e. if Z were on PQ produced, the above formulae would still apply but q would now be regarded as negative, in which case

$$x = \frac{px_2 - qx_1}{p-q} \quad \text{and} \quad y = \frac{py_2 - qy_1}{p-q}.$$

TYPICAL EXAMPLES

Ex. 1. If the point $(3, 4)$ is the middle point of the line joining the point $(6, 8)$ and another point (x, y), what are x and y?

3 must be the arithmetic mean of 6 and x,

4 must be the arithmetic mean of 8 and y,

$$\therefore\ 3 = \frac{x+6}{2} \quad \text{and} \quad 4 = \frac{y+8}{2};$$

whence $x = 0$ and $y = 0$, i.e. the other point is the origin.

Ex. 2. Given that the origin is the centroid of a triangle two of whose vertices are the points $(-5, -7)$, $(6, -4)$, find the coordinates of the other vertex.

Let the coordinates of the third vertex be (x, y).

Then we have

$$0 = \frac{x-5+6}{3}, \qquad 0 = \frac{y-7-4}{3};$$

whence $x = -1$, $y = 11$.

Ex. 3. P and Q are two fixed points and PM, QN of lengths h_1 and h_2 respectively are drawn perpendicular to a given straight line. If $MN = a$, find the area of the trapezium $PMNQ$.

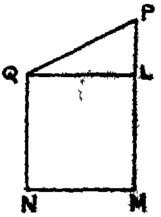
$$PL = PM - LM = PM - QN = h_2 - h_1,$$

$\therefore$ area of trapezium $PMNQ$

$$= \text{rectangle } QNML + \Delta PLQ$$

$$= h_2 a + \tfrac{1}{2}(h_1 - h_2)a$$

$$= \frac{a}{2}(h_1 + h_2).$$

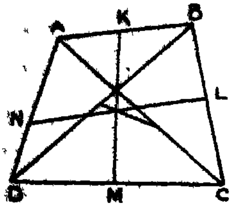


Ex. 4. $ABCD$ is a quadrilateral and K, L, M, N are the

middle points of its sides AB, BC, CD, DA respectively. Find

(1) the coordinates of the middle points of KM and LN;

(2) the coordinates of the middle point of the line joining the middle points of the diagonals of the quadrilateral $ABCD$.



Let the coordinates of A, B, C, D be $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ respectively.

K is the point $\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}$.

M is the point $\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}$.

$\therefore$ the coordinates of the middle point of KM are

$$\frac{x_1+x_2+x_3+x_4}{4}, \quad \frac{y_1+y_2+y_3+y_4}{4}.$$

Similarly, it will be found that the coordinates of the middle point of LN are

$$\frac{x_1+x_2+x_3+x_4}{4}, \quad \frac{y_1+y_2+y_3+y_4}{4},$$

which proves that KM and LN bisect each other and that $KLMN$ would therefore be a parallelogram.

Again, the middle point of AC is $\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}$, and that of BD is $\frac{x_2+x_4}{2}, \frac{y_2+y_4}{2}$.

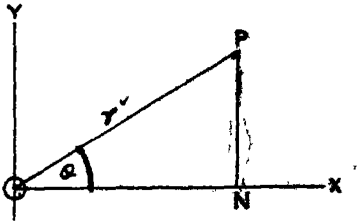
The middle point of the line joining the middle points of AC and BD is therefore the point whose coordinates are

$$\frac{x_1+x_2+x_3+x_4}{4}, \quad \frac{y_1+y_2+y_3+y_4}{4}.$$

Thus we have proved that the line joining the middle points of the diagonals of a quadrilateral passes through the point of intersection of the lines joining the middle points of its opposite [illegible] that these three lines are each bisected at this point.

4. The polar coordinates of a point

There is another way in which the position of a point can be fixed, viz. by ascertaining its distance from a fixed point called the *Pole* and the angle that the line joining it to the pole makes with a fixed straight line through the pole, usually called the *Initial Line*.



In the diagram O is the pole and OX the initial line. If $OP = r$ and

$$\angle POX = \theta,$$

then the polar coordinates of the point P are (r, θ).

θ, it may be noted, is always in an anti-clockwise direction.

Draw OY and PN perpendicular to OX and let PN meet OX in N.

Then, if (x, y) are the coordinates of P with regard to OX, OY as axes of coordinates,

$$\mathbf{x} = ON = OP\cos\angle PON = \mathbf{r}\cos\boldsymbol{\theta}$$

and $$\mathbf{y} = PN = OP\sin\angle PON = \mathbf{r}\sin\boldsymbol{\theta};$$

whence $$\mathbf{x}^2+\mathbf{y}^2 = \mathbf{r}^2 \quad\text{and}\quad \frac{\mathbf{y}}{\mathbf{x}} = \tan\boldsymbol{\theta}.$$

x and y are known as the *Cartesian Coordinates* of a point and the above equations show the relation between its cartesian and polar coordinates where the pole is the origin and the initial line the x-axis.

EXAMPLES 2

1. The extremities of a diameter of a circle are the points (3, 5), (−9, −5). What are the coordinates of the centre of the circle?

2. Find the coordinates of the middle points of the lines joining the following pairs of points: (0, 5), (3, 0); (6, 5), (−6, −5); (2, 1), (5, −4).

3. Find the coordinates of the points which divide the three

lines in the previous question in the ratio 3 : 2, (i) internally and (ii) externally.

4. Find the coordinates of the point which divides the line joining the points $(a+2b, a-2b)$, $(a-2b, a+2b)$ in the ratio $a : b$ internally.

5. O is a fixed point and the coordinates of a point A with regard to axes through O are $(3, 4)$; if $OA.OB = 15$, find the coordinates of B, a point on the line OAB.

6. Find the coordinates of the middle points of the diagonals of the quadrilateral whose vertices are $(0, 0)$, $(5, 3)$, $(6, -2)$, $(4, -3)$; also those of the middle point of the line which joins the two points when found.

7. *Find the area of the trapezium formed by the points* $(4, 3)$, $(4, -[illegible])$, $(2, 1)$, $(2, -2)$.

8. Show that the coordinates of the middle points of the lines joining the middle points of opposite sides of the trapezium in the last question are identical.

9. Find the distance between the two points which divide the line joining the points $(3, 4)$, $(6, -8)$ internally and externally in the ratio 2 : 1.

10. In what ratio does the point $(-2, 9)$ on the line joining the two points $(-3, 11)$, $(-1, 7)$ divide it internally, and in what ratio does the point $(2, 1)$ on the same line produced divide it externally?

11. Find the coordinates of the centroid of the triangle whose vertices are the points $(3, 4)$, $(2, -1)$, $(-4, 5)$.

12. The coordinates of the vertices of a triangle are $(3, 5)$, $(-2, 6)$, $(1, -1)$; find the coordinates of the centroid of the triangle formed by joining the middle points of its sides.

13. If the point $(3, 4)$ is the centroid of a triangle two of whose vertices are the points $(6, 2)$, $(-5, -4)$, what are the coordinates of the remaining vertex?

14. Find the length of the line joining two points whose polar coordinates are (r_1, θ_1), (r_2, θ_2).

15. Find the area of a triangle whose polar coordinates are [illegible], 90°), (9, 120°).

Formulae of Chapter I for reference

1. The distance between two points (x_1, y_1), (x_2, y_2) is

$$\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2\}}.$$

2. The coordinates of the middle point of the line joining the two points (x_1, y_1), (x_2, y_2) are

$$\frac{x_1+x_2}{2}, \quad \frac{y_1+y_2}{2}.$$

3. The coordinates of a point dividing the line joining the two points (x_1, y_1), (x_2, y_2) internally and externally in the ratio $p:q$ are

$$\frac{px_2 \pm qx_1}{p \pm q}, \quad \frac{py_2 \pm qy_1}{p \pm q}.$$

4. The coordinates of the centroid of a triangle whose vertices are the points (x_1, y_1), (x_2, y_2), (x_3, y_3) are

$$\frac{x_1+x_2+x_3}{3}, \quad \frac{y_1+y_2+y_3}{3}.$$

5. The relation connecting cartesian and polar coordinates of a point is

$$x = r\cos\theta, \quad y = r\sin\theta.$$

II

STRAIGHT LINES

As was pointed out in the Introduction, a plane drawn through a generator of a conical surface will generally intersect the surface again in another generator. Hence two straight lines are a conic section.

These two lines coincide and become one straight line when the intersecting plane is a tangent plane to the cone.

1. The linear equation

The most general form in algebra of an equation of the first degree in x and y is $ax+by+c = 0$, where x and y are the variables and a, b, and c are constants, i.e. quantities which always remain the same.

In this equation $ax+by+c = 0$, if x is assigned any particular value, then one, and only one, corresponding value of y can be obtained to satisfy the equation. Therefore we may say that y is dependent upon x, or x dependent upon y, the main point being that they are not independent of each other if they are connected by the equation.

Consider the equation $3x+4y = 12$.

Give x a number of values in turn such as

$$x = -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3.$$

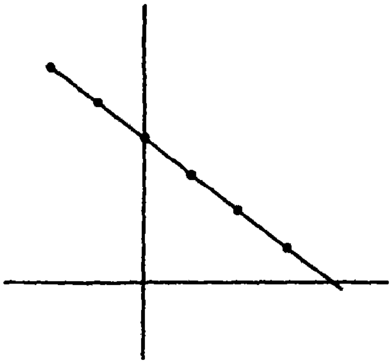
By putting these values of x in the equation we find that

$$y = 5\tfrac{1}{4},\ 4\tfrac{1}{2},\ 3\tfrac{3}{4},\ 3,\ 2\tfrac{1}{4},\ 1\tfrac{1}{2},\ \tfrac{3}{4}.$$

Now take a piece of graph paper and plot the points $(-3, 5\tfrac{1}{4})$, $(-2, 4\tfrac{1}{2})$, $(-1, 3\tfrac{3}{4})$, $(0, 3)$, $(1, 2\tfrac{1}{4})$, $(2, 1\tfrac{1}{2})$, $(3, \tfrac{3}{4})$ with respect to a pair of axes at right angles conveniently chosen.

If the points have been accurately plotted it will now be found that they all lie on one straight line. Draw it, and then the coordinates of any point on it can at once be obtained by simple measurement. There are an unlimited number of points on the line each with its x and y coordinates satisfying the equation $3x+4y = 12$.

It would also be found that if the coordinates of any point not on the line were measured, the values of x and y so obtained would not satisfy the equation.



It would seem, then, by this method of trial by drawing, that every equation of the first degree represents a straight line. We shall now give a formal proof of this.

Suppose that the coordinates of two points P and Q, (x_1, y_1), (x_2, y_2) respectively, satisfy the equation $ax+by+c = 0$.

Also that the coordinates (X, Y) of some third point (R) satisfy it.

Then, if we prove that the three points P, Q, R are collinear, the theorem is established.

Let the ordinates of P, Q, R be PL, QM, RN respectively, and through P draw a straight line parallel to the x-axis cutting QM in m and RN in n, or these lines produced in m and n.

Then, by data, $ax_1+by_1+c = 0$ and $ax_2+by_2+c = 0$,

$$\therefore\ a(x_1-x_2)+b(y_1-y_2) = 0, \qquad \therefore\ \frac{y_1-y_2}{x_1-x_2} = -\frac{a}{b}.$$

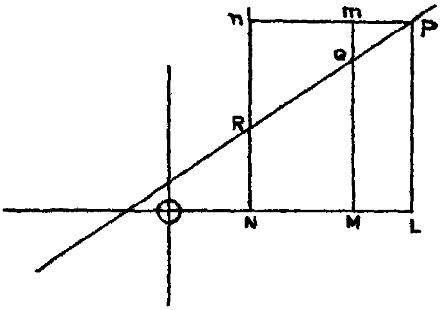
Again, $$aX+bY+c = 0.$$

And in the same way

$$\frac{y_1-Y}{x_1-X}=-\frac{a}{b}=\frac{y_1-y_2}{x_1-x_2}.$$

Hence, as in the figure,

$$\frac{Rn}{Pn}=\frac{Qm}{Pm}.$$



$\therefore$ the triangles RnP, QmP are similar.

$\therefore \angle nPR=\angle mPQ$, $\quad \therefore$ P, Q, R are collinear.

Hence any point whose coordinates satisfy the equation $ax+by+c=0$ must always lie on one and the same straight line which the equation therefore represents.

The reason can now be seen why any equation of the first degree in x and y is called a *Linear Equation*, because it must represent a straight line if it be graphically interpreted.

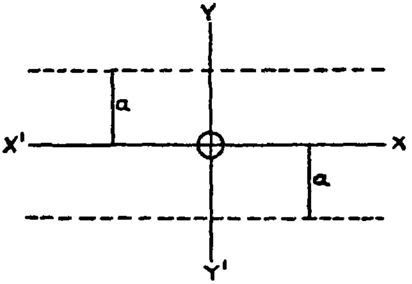
2. Equations of lines parallel to either axis

Take two straight lines drawn parallel to the x-axis and at a distance a from it either above or below.

In the former case the ordinate of any point on the line must be $+a$, and in the latter case $-a$.

That is to say, any point on the former line satisfies the equation $y=a$, and any point on the latter line satisfies the equation $y=-a$.

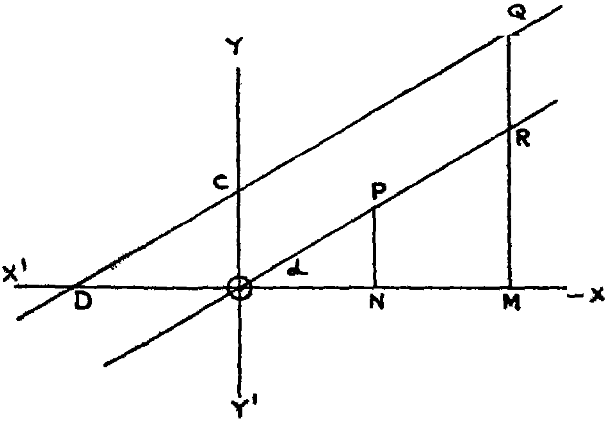
Similarly, lines parallel to the y-axis and at a distance b from it will have as their equations $x = b$ or $x = -b$ according as to whether the line is to the right or to the left of the y-axis.



The equations of the axes, it may be noted, are $\mathbf{y = 0}$ and $\mathbf{x = 0}$.

3. Equation of a straight line through the origin

Let a straight line OP make an angle $POX = \alpha$, with the x-axis. From any point $P\ (x, y)$ on it draw PN perpendicular to the x-axis.



Then
$$\tan\alpha = \frac{PN}{ON} = \frac{y}{x},$$

which may be written $\quad y = x\tan\alpha,$

or $\quad \mathbf{y = mx}, \quad$ **where** $\quad \mathbf{m = \tan\alpha}.$

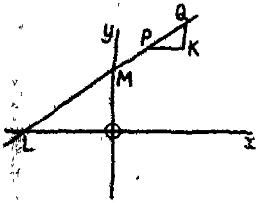
The angle which a straight line makes with the x-axis, reckoning opposite to the motion of the hands of a watch, is called the *slope* of the line.

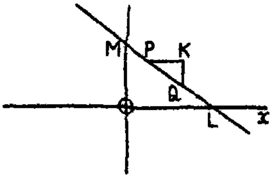
In the equation of the line $y = mx$ we have seen that $m = \tan\alpha$, where α is the slope of the line.

Suppose that P and Q are two points on any straight line which cuts the axes in L and M respectively.

Draw PK, QK parallel to the axes, to meet in K.

Then, paying due regard to sign, and assuming that the same scales are used for measuring the x and y coordinates, the *gradient* of the line is defined as the change in y per unit change in x as a point travels along the line.





The *two* figures illustrate the cases where the slope of the line is acute and obtuse.

In the first case the change for PK in x is QK in y. Therefore the gradient, which is the change in y for a unit change of x, is

$$\frac{QK}{PK} = \tan\angle QPK = \tan\angle PLx.$$

In the second case the gradient is

$$-\frac{QK}{PK} = -\tan\angle QPK = -\tan(180^\circ - \angle PLx) = \tan\angle PLx.$$

Hence in either case *the tangent of the angle of slope of a line is equal to its gradient.*

It follows that the equation of a line being $y = mx$, its gradient is m.

Obviously lines with the same gradient are parallel.

4. The linear equation $\mathbf{y = mx+c}$

Taking the first diagram of the previous article, let QCD be a straight line parallel to OP and let it cut the axes of y and x in C and D respectively. Let $OC = c$.

Now let Q be any point (x, y) on this line, and from it draw QRM perpendicular to the x-axis, cutting it in M and the line OP, produced if necessary, in R.

Then $\angle QDX = \angle ROX = \alpha$, and $\tan\alpha = m$.

$$\therefore\ m.OM = OM\tan\alpha = RM = QM-QR = QM-OC,$$

$$\therefore\ mx = y-c, \quad\text{or}\quad \mathbf{y = mx+c},$$

which is the equation of the straight line whose gradient is m and which makes an intercept c on the y-axis.

The general equation of the first degree $ax+by+c = 0$ can be rewritten in the form

$$y = -\frac{a}{b}x-\frac{c}{b},$$

which shows at once by inspection that its gradient is $-\frac{a}{b}$ and its intercept on the y-axis is $-\frac{c}{b}$.

The gradient of this line could also be obtained by differentiating the equation $ax+by+c = 0$, when we should have

$$a+b\frac{dy}{dx} = 0,$$

whence $$\frac{dy}{dx} = -\frac{a}{b}.$$

The most simple way, therefore, of getting the gradient of any straight line is by differentiating its equation. But to those unfamiliar with the calculus a useful rule for doing this is the following:

(1) Arrange the x and y terms on opposite sides of the equation.

(2) Divide the coefficient of x by that of y.

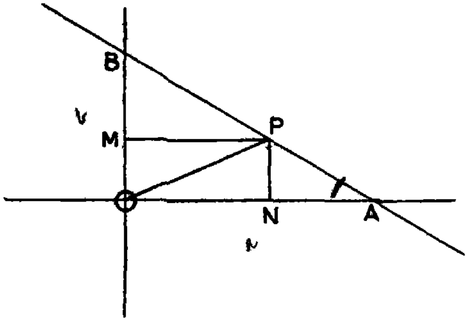
Thus, according to this rule, if we are dealing with a linear equation such as $3x+2y-7 = 0$, by (1) $2y = -3x+7$, by (2) the gradient is $-\frac{3}{2}$.

5. The 'intercept' equation $\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}} = 1$

Suppose that a straight line cuts the x-axis in A and the y-axis in B. Let the intercept OA on the x-axis be a and that on the y-axis be b.

Let P be any point (x, y) on the line AB.

From P draw perpendiculars PM, PN to the axes. Join OP.



From the diagram it is seen that

$$\Delta AOB = \Delta AOP+\Delta POB,$$

$$\therefore\ \tfrac{1}{2}OA.OB = \tfrac{1}{2}PN.OA+\tfrac{1}{2}PM.OB,$$

$$\therefore\ ab = ya+xb.$$

$$\therefore\ \frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}} = 1.$$

which is the equation of a straight line whose intercepts on the axes are a and b.

If $l = \frac{1}{a}$ and $m = \frac{1}{b}$, this equation can be written

$$\mathbf{lx+my = 1},$$

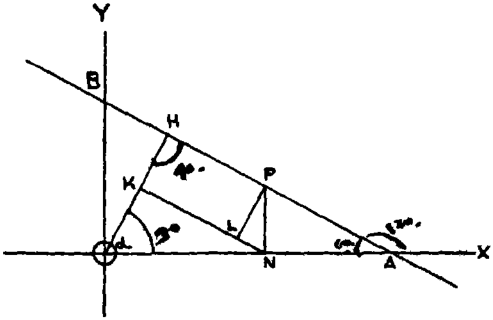
l and m being the reciprocals of the intercepts on the axes made by the line.

6. The equation $\mathbf{x \cos \alpha + y \sin \alpha = p}$

Let a straight line cut the axes in A and B. Through the origin O draw OH perpendicular to AB and let it make an angle α with the x-axis. Let $OH = p$.

Take any point $P\ (x, y)$ on the line AB.

Draw PN perpendicular to the x-axis, NK perpendicular to OH, and PL perpendicular to NK.



Then $\quad p = OH = OK + KH = OK + PL.$

But

$$\angle LPN = 90^\circ - \angle LNP = \angle KNO = 90^\circ - \angle KON = 90^\circ - \alpha.$$

$$\begin{aligned} \therefore\ p &= ON \cos \alpha + PN \cos LPN \\ &= ON \cos \alpha + PN \cos(90^\circ - \alpha) \\ &= x \cos \alpha + y \sin \alpha. \end{aligned}$$

$\therefore$ the equation of the line is

$$\mathbf{x \cos \alpha + y \sin \alpha = p},$$

where p is the length of the perpendicular from the origin on it, and α the angle this perpendicular makes with the x-axis.

It is important to notice that any one of the five standard forms of the equation of a straight line we have now obtained, viz. (1) $y = mx+c$, (2) $Ax+By+C = 0$, (3) $\frac{x}{a}+\frac{y}{b} = 1$, (4) $lx+my = 1$, (5) $x\cos\alpha+y\sin\alpha = p$, can be changed into any one of the others.

Take, for example, the equation $Ax+By+C = 0$.

Rewrite it $y = -\frac{A}{B}x-\frac{C}{B}$ and we have it in the first form.

Rewrite it
$$\frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}} = 1$$
and we have it in the third form.

Or as $-\frac{A}{C}x-\frac{B}{C}y = 1$ it is in the fourth form.

To identify it with the fifth form let
$$r\cos\alpha = A, \qquad r\sin\alpha = B.$$

Then
$$\tan\alpha = \frac{B}{A} \quad\text{and}\quad r^2 = A^2+B^2.$$

The equation becomes $r\cos\alpha x+r\sin\alpha y+C = 0$, i.e.
$$x\cos\alpha+y\sin\alpha = -\frac{C}{r} = -\frac{C}{\sqrt{(A^2+B^2)}},$$
which is of the form $x\cos\alpha+y\sin\alpha = p$.

Incidentally, we see from this that, if the equation of a straight line be $Ax+By+C = 0$, the perpendicular from the origin on it has a gradient of $\frac{B}{A}$ and its length is $\frac{C}{\sqrt{(A^2+B^2)}}$.

The student should memorize the above five standard forms of the equation of a straight line and satisfy himself that he can give a geometrical interpretation to the various letters which occur in them.

TYPICAL EXAMPLES

Ex. 1. What inferences may be drawn from inspection of the equation $ax-by=0$?

Firstly, $x=0$, $y=0$ clearly satisfies it. Therefore it is a line going through the origin.

Secondly, its gradient is $\frac{a}{b}$.

Ex. 2. Write down the equation of the straight line which is distant 4 from the origin and makes an angle of 120° with the x-axis.

Using the standard form $x\cos\alpha+y\sin\alpha=p$.

Here $p=4$ and $\alpha=30°$.

$\therefore$ the required equation is $x\cos 30°+y\sin 30°=4$, i.e.

$$x\sqrt{3}+y=8.$$

Ex. 3. Find the equation of a straight line parallel to $10x+24y=65$ but twice as far away from the origin.

Rewriting the equation

$$\frac{10x}{\sqrt{(10^2+24^2)}}+\frac{24y}{\sqrt{(10^2+24^2)}}=\frac{65}{\sqrt{(10^2+24^2)}},$$

i.e.
$$\frac{10x}{26}+\frac{24y}{26}=\frac{65}{26}.$$

The required equation is clearly

$$\frac{10x}{26}+\frac{24y}{26}=\frac{130}{26},$$

i.e.
$$5x+12y=65.$$

Ex. 4. Show that the lines $3x+2y=5$ and $2x+3y=7$ make equal angles with the axes of x and y respectively.

If α be the angle the first line makes with the x-axis, then $\tan\alpha=-\frac{3}{2}$.

If β be the angle the second line makes with the x-axis, then $\tan\beta=-\frac{2}{3}$.

This line makes an angle of $90°-\beta$ with the y-axis.

Now $\tan\alpha =$ [illegible] $= \cot\beta = \tan(90^\circ-\beta)$,

$$\therefore\ \alpha = 90^\circ-\beta.$$

Ex. 5. Make a graph of the line $3x-4y+12 = 0$.

Writing the equation $\frac{x}{-4}+\frac{y}{3} = 1$, we see that it makes intercepts of -4 and 3 on the axes.

Hence plot the points $(-4, 0)$, $(0, 3)$ and join them.

N.B. The quickest way of plotting a line whose equation is given is by finding its intercepts on the axes and so obtaining two points on it which determine the line.

EXAMPLES 3

1. Does the line whose equation is $7x-5y = 12$ pass through any of the following points: $(2, 1)$, $(6, 6)$, $(0, 0)$, $(-5, 7)$, $(2, \frac{2}{5})$?

2. Give a reason why, if c is not zero, the line $ax+by+c = 0$ cannot pass through the origin.

3. Draw graphs of the lines (1) $2x+5y = 7$, (2) $x+4y = 3$, (3) $y = 2x-5$.

4. Find the equation of the straight line such that the perpendicular from it to the origin is of length 3 and which makes an angle of 60° with the axis of x.

5. Find the equation of the line which makes an angle of 60° with the x-axis, and an intercept of -2 on the y-axis.

6. Find the equation of the line which makes an angle of 120° with the x-axis, and an intercept of $\sqrt{3}$ on the y-axis.

7. Find the equation of a straight line which makes an intercept of 5 between the axes and an angle of 120° with the positive direction of the x-axis.

8. If the line whose equation is $\frac{x}{a}+\frac{y}{b} = 1$ passes through the point $(-3, 3)$, what relation exists between a and b?

9. What are the gradients of, and intercepts [illegible] by, the lines whose equations are (1) $5x+7y = 9$, (2) [illegible] (3) $4x+5y = 21$?

10. Show that the line [illegible] the points [illegible] [illegible]sected by the line wh[illegible] [illegible]

11. If the line $ax+3y = 4$ is parallel to the line $2y+3x = 5$, what is a?

12. Put the equation $5x-12y+13 = 0$ in the form

$$x\cos\alpha+y\sin\alpha = p;$$

hence determine the length of the perpendicular from the origin on the line.

13. If the line $ax+4y = 5$ makes an intercept of 3 on the x-axis, find a.

14. The straight line $y = mx+c$ is not parallel to the line whose equation is $4x-3y+5 = 0$, but the two lines make equal angles with the x-axis. What is m?

15. What is the perpendicular distance between the two parallel lines $3x-4y = 7$ and $3x-4y = 10$?

16. Show that the intercepts made on the axes by the line whose equation is $ax+by+c = 0$ can be obtained by putting $x = 0$ or $y = 0$ in the equation.

17. Interpret the equation $0.x+0.y = c$.

18. Find the equation of the line terminated at each extremity by the axes which is bisected at the point $(2, 3)$.

7. The equation $y-k = m(x-h)$

This equation is clearly of the first degree in x and y, therefore it represents some straight line.

Again, $x = h$, $y = k$ clearly satisfy the equation. Therefore the straight line represented by it must pass through the point (h, k).

Finally its gradient is m.

∴ we conclude that the equation $y-k = m(x-h)$ represents a straight line of gradient m and passing through the point (h, k).

8. The equation of the straight line joining two points (x_1, y_1), (x_2, y_2)

The equation of any straight line through the point (x_1, y_1) can be written $y-y_1 = m(x-x_1)$.

If this passes through the point (x_2, y_2), we have

$$y_2-y_1 = m(x_2-x_1).$$

Dividing, $$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2},$$

which is the equation of the straight line joining the two points (x_1, y_1), (x_2, y_2).

This result is so often required that it should be carefully memorized.

Note that the equation of the line joining the origin to the point (x_1, y_1) would be

$$\frac{x-x_1}{x_1-0} = \frac{y-y_1}{y_1-0},$$

which reduces to $$\frac{x}{x_1} = \frac{y}{y_1}.$$

9. The point of intersection of two straight lines

Let the two straight lines be

$$ax+by+c = 0, \qquad a'x+b'y+c' = 0.$$

Then finding the point of intersection is the same thing as finding the values of x and y which satisfy both equations, i.e. it is obtained by solving them, when we have

$$\frac{x}{bc'-b'c} = \frac{y}{ca'-c'a} = \frac{1}{ab'-a'b}.$$

If three straight lines $ax+by+c = 0$, $a'x+b'y+c' = 0$, $a''x+b''y+c'' = 0$ are concurrent, i.e. if all three meet in the same point, the coordinates of the point in which the first two, say, intersect, must satisfy the third equation.

Hence $$a''\frac{bc'-b'c}{ab'-a'b}+b''\frac{ca'-c'a}{ab'-a'b}+c'' = 0;$$

whence $$a''(bc'-b'c)+b''(ca'-c'a)+c''(ab'-a'b) = 0,$$

which is the required condition for the concurrency of the three lines.

The student familiar with the determinant notation will see that this result can be expressed

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0,$$

a form which is easily memorized.

10. The equation $ax+by+c+\lambda(a'x+b'y+c') = 0$

Using the same method as that of Article 7, which we may call the *retrospective* method, we observe:

(1) that the equation is of the first degree in x and y and therefore that of some straight line;

(2) that, if any values of x and y satisfy both $ax+by+c = 0$ and $a'x+b'y+c' = 0$, they must satisfy the equation we are considering.

The coordinates of the point of intersection of the two lines $ax+by+c = 0$, $a'x+b'y+c' = 0$ therefore satisfy the equation $ax+by+c+\lambda(a'x+b'y+c') = 0$.

Hence this equation is that of a straight line passing through the point of intersection of the lines whose equations are $ax+by+c = 0$, $a'x+b'y+c' = 0$.

λ, it is hardly necessary to add, can have any value positive or negative, since an infinite number of lines can be drawn through the point of intersection of the two given lines.

11. The length of the perpendicular from a given point (h, k) to a given straight line $ax+by+c = 0$ is numerically of a length

$$\frac{ah+bk+c}{\sqrt{(a^2+b^2)}}.$$

Let the given straight line cut the axes in the points L and M. From P the point (h, k) draw perpendiculars PH, PK to the axes and PN perpendicular to the line LM.

Join PL, PM, PO.

This equation could also be thus obtained:

The cartesian equation of the line is

$$x\cos\alpha+y\sin\alpha=p.$$

Putting $\qquad x=r\cos\theta,\qquad y=r\sin\theta,$

we have $\qquad r\cos\theta\cos\alpha+r\sin\theta\sin\alpha=p;$

whence $\qquad r\cos(\theta-\alpha)=p.$

13. Area of a triangle

Suppose that the three vertices of a triangle are the points (x_1, y_1), (x_2, y_2), (x_3, y_3).

The equation of the line joining the points (x_2, y_2), (x_3, y_3) is

$$\frac{x-x_2}{x_2-x_3}=\frac{y-y_2}{y_2-y_3},$$

i.e. $$x(y_2-y_3)-y(x_2-x_3)+x_2y_3-x_3y_2=0.$$

The length of the perpendicular from (x_1, y_1) on this line is

$$\frac{x_1(y_2-y_3)-y_1(x_2-x_3)+x_2y_3-x_3y_2}{\sqrt{\{(x_2-x_3)^2+(y_2-y_3)^2\}}}.$$

But the distance between the points (x_2, y_2), (x_3, y_3) is

$$\sqrt{\{(x_2-x_3)^2+(y_2-y_3)^2\}}.$$

The area of the triangle is half the product of these two last results, viz. $\frac{1}{2}(x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3)$, or in determinant notation:

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Typical Examples

Ex. 1. What is the equation of the straight line through the point (2, −1) which makes an angle of 60° with the x-axis?

The equation of any line through $(2, -1)$ is

$$y+1=m(x-2).$$

Here $m=\tan 60^\circ=\sqrt{3}$.

$\therefore$ the required equation is $y+1=\sqrt{3}(x-2)$

Ex. 2. Find the equation of the line joining the points (a, b), (b, a), and that of the line through the point $(a+b, a-b)$ parallel to it.

The equation of the line joining (a, b), (b, a) is

$$\frac{x-a}{a-b}=\frac{y-b}{b-a}, \checkmark$$

i.e.
$$x+y=a+b, \checkmark$$

of which the gradient is -1. ✓

The line through $(a+b, a-b)$ ✓ parallel to this line is

$$y-(a-b) = -1[x-(a+b)],$$

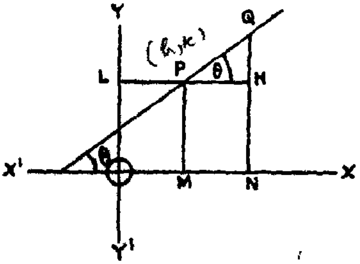
i.e.
$$x+y = 2a.$$

Ex. 3. Show that, if (h, k) be a fixed point on a straight line making an angle θ with the x-axis, the coordinates of a point on the line distant r from the point (h, k) are

$$h+r\cos\theta, \quad k+r\sin\theta.$$

and also that the equation of the line may be written

$$\frac{x-h}{\cos\theta}=\frac{y-k}{\sin\theta}=r.$$



Let PQ be the line and P the given point (h, k) on it.

Draw PM, QN perpendicular to the x-axis and PL perpendicular to the y-axis; produce LP to cut QN in H.

Let $PQ = r$ and $\angle QPH = \theta$.

The abscissa of Q is clearly

$$ON = OM+MN = OM+PH = h+r\cos\theta.$$

And the ordinate of Q is

$$QN = NH+HQ = MP+HQ = k+r\sin\theta.$$

So if Q be the point (x, y) and PQ is r for that point, we have

$$x = h+r\cos\theta, \qquad y = k+r\sin\theta.$$

Hence $$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r.$$

Ex. 4. Find the equation of the straight line which passes through the point of intersection of the lines $3x-5y = 4$ and $2x+3y = 28$, and also through the point $(1, -1)$.

Any line through the point of intersection of the two given straight lines may have its equation written

$$3x-5y-4-\lambda(2x+3y-28) = 0.$$

If $x = 1$, $y = -1$ satisfy this equation, we have

$$3+5-4-\lambda(2-3-28) = 0. \quad \therefore \lambda = -\tfrac{4}{29}.$$

$\therefore$ the required equation is

$$29(3x-5y-4)+4(2x+3y-28) = 0,$$

i.e. $$95x-133y = 228.$$

Ex. 5. Prove by using the formula for the area of a triangle that the general equation of the first degree must represent a straight line.

Suppose that three points (x_1, y_1), (x_2, y_2), (x_3, y_3) satisfy the equation $ax+by+c = 0$.

Then

$$ax_1+by_1+c = 0, \qquad ax_2+by_2+c = 0, \qquad a_3x+b_3y+c = 0.$$

Hence, as in Art. 9,

$$x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3 = 0.$$

But the expression on the left-hand side of this equation is twice the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3).

$\therefore$ the area of this triangle is zero.

$\therefore$ the three points must lie on one straight line.

$\therefore$ [illegible] $+by+c = 0$ must represent a straight line, seeing that [illegible] points taken on it form a triangle of zero area.

Ex. 6. Find the equations of the diagonals of the parallelograms formed by the lines

$$4x-5y = 7,\ 4x-5y = 14;\ 3x+7y = 8,\ 3x+7y = 12.$$

Any line through the intersection of the first and third of these lines can have its equation written

$$4x-5y-7-\lambda(3x+7y-8) = 0.$$

So any line through the intersection of the second and fourth of these lines can also have its equation written

$$4x-5y-14-\mu(3x+7y-12) = 0.$$

If these two are one and the same line, i.e. a diagonal, then the equations

$$(4-3\lambda)x-(5+7\lambda)y-7+8\lambda = 0$$

and $$(4-3\mu)x-(5+7\mu)y-14+12\mu = 0$$

must be identical, therefore

$$\frac{4-3\lambda}{4-3\mu} = \frac{5+7\lambda}{5+7\mu} = \frac{7-8\lambda}{14-12\mu};$$

whence $$\lambda = \mu = \tfrac{7}{4}$$

and the equation of the diagonal is

$$4(4x-5y-7) = 7(3x+7y-8),$$

i.e. $$5x+69y = 28.$$

Similarly, the equation of the other diagonal will be

$$4(4x-5y-7)+7(3x+7y-12) = 0,$$

i.e. $$37x+29y = 112.$$

This question could also be done by solving the equations and finding the vertices of the parallelogram, but the 'method of identification' is instructive and often employed later.

Ex. 7. Show that the area of the triangle, whose vertices are the points

$$(x_2+x_3-3x_1,\ y_2+y_3-3y_1),\quad (x_3+x_1-3x_2,\ y_3+y_1-3y_2),$$
$$(x_1+x_2-3x_3,\ y_1+y_2-3y_3),$$

is 16 times the area of the triangle Δ whose vertices are the points (x_1, y_1), (x_2, y_2), (x_3, y_3). [C.]

Let $p = x_1+x_2+x_3$ and $q = y_1+y_2+y_3$.

Then the coordinates of the vertices of the required triangle are $(p-4x_1, q-4y_1)$, $(p-4x_2, q-4y_2)$, $(p-4x_3, q-4y_3)$. And the area of the triangle is

$$\begin{aligned}
&(p-4x_2)(q-4y_3)-(p-4x_3)(q-4y_2)\\
&\quad+(p-4x_3)(q-4y_1)-(p-4x_1)(q-4y_3)\\
&\qquad+(p-4x_1)(q-4y_2)-(p-4x_2)(q-4y_1)\\
&\qquad\quad= -4p(y_3-y_2+y_1-y_3+y_2-y_1)\\
&\qquad\qquad-4q(x_2-x_3+x_3-x_1+x_1-x_2)\\
&\qquad\qquad\quad+16(x_2y_3-x_3y_2+x_3y_1-x_1y_3+x_1y_2-x_2y_1)\\
&\qquad\quad= 16\Delta.
\end{aligned}$$

Students familiar with determinants will see that

$$2\text{ (area of triangle)} = \begin{vmatrix} p-4x_1 & q-4y_1 & 1 \\ p-4x_2 & q-4y_2 & 1 \\ p-4x_3 & q-4y_3 & 1 \end{vmatrix}$$

which splits up into the four determinants

$$\begin{vmatrix} p & q & 1 \\ p & q & 1 \\ p & q & 1 \end{vmatrix} -4q\begin{vmatrix} x_1 & 1 & 1 \\ x_2 & 1 & 1 \\ x_3 & 1 & 1 \end{vmatrix} -4p\begin{vmatrix} 1 & y_1 & 1 \\ 1 & y_2 & 1 \\ 1 & y_3 & 1 \end{vmatrix} +16\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

of which all but the last are zero, having two columns identical.

$\therefore$ area of triangle $= 16\Delta$, as before.

EXAMPLES 4

1. Find the equation of the lines joining the following pairs of points:

$(0, 0)$, $(3, -4)$; $(2, -5)$, $(-3, -4)$; $(a+b, a+2b)$, $(a-b, a-2b)$.

2. Find the equation of the line through the point [illegible] which makes an angle of 30° with the x-axis.

3. Find the equation of a line through the point (a, b) which is parallel to the line $ax+by+c = 0$.

✓4. Find the point of intersection of the lines $lx+my = 1$ and $l'x+m'y = 1$.

5. Find the coordinates of the point of intersection of the following lines:

✓(1) $3x+5y = 11, x-2y = 4$.
✓(2) $ax+by+c = 0, bx-ay-c = 0$.
(3) $x\cos\alpha+y\sin\alpha = p, x\cos\alpha'+y\sin\alpha' = p'$.

6. Find the equation of the line through the point $(4, -2)$ which makes equal intercepts on the axes.

7. Find the equation of the straight line which joins the point of intersection of the lines $3x-4y = 8$ and $7x-5y = 23$ to the origin.

8. Find the equation of the line joining the point (a, b) to the point of intersection of the lines $\frac{x}{a}+\frac{y}{b} = 1$ and $\frac{x}{b}+\frac{y}{a} = 1$.

9. Show that the three lines $2x+3y = 1$, $3x+2y = 1$, $y = x$, meet at a point.

10. Without using any formula, show that the three lines $2x-5y = 6$, $3x+4y+14 = 0$, $5x-7y-4 = 0$ are concurrent.

11. Find the perpendicular distance from the point $(2, 3)$ of the following lines: (1) $x+y = 1$, (2) $3x-4y = 20$, (3) $5x+12y = 26$, (4) $x\sqrt{3}+y = 2$.

12. Find the equation of the straight lines through the point $(3, 2)$ which will make with the x-axis an isosceles triangle whose base on that axis is 5.

13. What is the perpendicular distance between the two parallel lines $y = mx+c$, $y = mx+d$?

14. Which of the following points lie on the same side of the line $3x-y-4 = 0$ as the origin, and which do not: $(0, 4)$, $(4, 1)$, $(5, -2)$, $(2, 2)$, $(-1, -2)$?

15. Without using any formula, find the area of the triangles formed by the three points $(0, 0)$, $(3, 4)$, $(-2, -2)$, and by the three points $(1, -1)$, $(2, 3)$, $(5, 6)$.

16. Find the equation of the line which joins the middle points of the line joining the points (a, b), (c, d), and of that joining the points (a', b'), (c', d').

17. Find the equations of the bisectors of the angles of a triangle whose vertices are the points $(0, 0)$, $(4, 0)$, $(0, 3)$, and prove that they meet in a point.

18. Find the equations of the medians of the triangle whose vertices are the points $(0, 0)$, (a, b), (a', b').

19. Show that the equation $4x+8y-3-\lambda(2x+3y-5) = 0$ represents a straight line which passes through the point $(7\frac{3}{4}, -3\frac{1}{2})$.

20. Show that the line whose equation is

$$(1-3\kappa)x+(2-\kappa)y-4-3\kappa = 0$$

always passes through a fixed point, no matter what the value of κ may be; find also the coordinates of this point.

21. Find the equation of the line parallel to the line whose equation is $2x+3y = 0$ which passes through the point of intersection of the lines whose equations are

$$x-2y-5 = 0, \qquad x+3y-10 = 0.$$

22. Find the equation of the line which joins the origin to the point of intersection of the lines whose equations are $lx+my = 1$ and $l'x+m'y = 1$.

23. Find the equation of the line which joins the point $(3, -2)$ to the point of intersection of the lines whose equations are

$$4x-7y+5 = 0, \qquad 2x-5y+11 = 0.$$

24. Find the equations of the lines joining the following pairs of points:

(1) $(a\cos\alpha, a\sin\alpha)$, $(a\cos\beta, a\sin\beta)$.

(2) $(at^2, 2at)$, $(at'^2, 2at')$.

(3) $(a\cos\theta, b\sin\theta)$, $(a\cos\phi, b\sin\phi)$.

(4) $\left(c\mu, \frac{c}{\mu}\right)$, $\left(c\mu', \frac{c}{\mu'}\right)$.

25. Find the length of the perpendiculars from the origin on the lines:

(1) $x\cos\theta+y\sin\theta = a$.

(2) $x-ty+at^2 = 0$.

(3) $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi = 1$.

(4) $x+y\mu^2 = 2c\mu$.

26. Find the coordinates of the point of intersection of the lines whose equations are:

(1) $x-yt_1+at_1^2 = 0$, $x-yt_2+at_2^2 = 0$.

(2) $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta = 1,\ \frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi = 1.$

(3) $x+y\mu_1^2 = 2c\mu_1,\ x+y\mu_2^2 = 2c\mu_2.$

27. Find the equations of the diagonals of the parallelogram formed by the lines $y = mx$, $y = mx+c$, $y = nx$, $y = nx+d$.

28. Interpret the polar equation $r\cos\theta = c$.

29. Find the polar and cartesian coordinates of the point of intersection of the lines whose polar equations are $r\cos\theta = c$ and $r\cos(\theta-\alpha) = d$.

30. Show that the length of the straight line drawn from the point (h, k) to the straight line $ax+by+c$ in a direction making an angle of θ with the x-axis is

$$\frac{ah+bk+c}{a\cos\theta+b\sin\theta}.$$

31. Find the equation of the diagonals of a rectangle which has sides of length a and b, one vertex at the origin, and one of its sides there making an angle of α with the x-axis.

32. Show that the origin lies within the triangle formed by the three lines whose equations are:

(1) $8x+3y = 24$, (2) $3x-4y = 6$, (3) $2x-y+4 = 0$.

33. Find the polar equation of the line which passes through two points whose polar coordinates are (μ_1, θ_1), (μ_2, θ_2).

34. The equations of four straight lines EAB, BCF, CDE, FDA are $3x-2y+1 = 0$, $4x+y-3 = 0$, $x+3y+2 = 0$, and $3(3x-2y+1)-2(4x+y-3)+(x+3y+2) = 0$, respectively. Obtain, preferably without finding the coordinates of the points A, B, C, D, E, F, the equations of the three straight lines BD, AC, EF. [O.]

35. The vertices A, B, C of a triangle have coordinates $(1, 3)$, $(-2, 1)$, $(3, 2)$. The line joining A to the origin meets BC at D. Find the ratio $BD : DC$. [N.]

36. Find the area of the triangle whose vertices are the points (r_1, θ_1), (r_2, θ_2), (r_3, θ_3) in polar coordinates.

37. Find the polar coordinates of the point of intersection of the lines whose equations are

$$\frac{l}{r} = \cos\theta-\cos(\theta-\alpha) \quad \text{and} \quad \frac{l}{r} = \cos\theta-\cos(\theta-\beta).$$

38. Show that the equation $r^2\cos^2(\theta-\alpha) = p^2$ represents a pair of parallel straight lines.

39. Find the polar equation of the line joining the points $(3\cos\alpha, 2\alpha)$ and $(3\cos 2\alpha, 3\alpha)$.

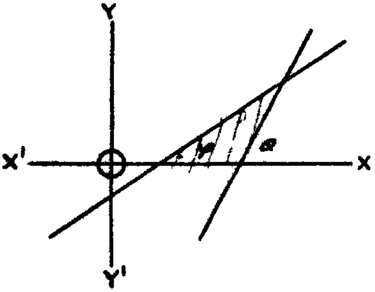
40. Find the area of the portion of the triangle formed by the lines $x-6y+14 = 0$, $2x-y-5 = 0$, $3x+4y-2 = 0$, which lies in the first quadrant.

14. The angle between two straight lines

Suppose that the equations of two straight lines are $y = mx+c$ and $y = m'x+c'$, and that they make angles of θ and ϕ with the x-axis.

Then $\tan\theta = m, \quad \tan\phi = m'.$

$$\therefore\ \tan(\theta-\phi) = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi} = \frac{m-m'}{1+mm'}.$$



As appears from the diagram, the acute angle between the two straight lines is $\theta-\phi$.

$\therefore$ the angle between them is

$$\tan^{-1}\frac{m-m'}{1+mm'}.$$

When speaking of the angle between two straight lines, the acute angle is usually meant unless anything is stated to the contrary.

Two things should be noted about the result just obtained.

Firstly, if $m = m'$, $\tan(\theta-\phi) = 0$, and $\theta = \phi$, i.e. the two straight lines are parallel.

Secondly, if $1+mm' = 0$, i.e. if $mm' = -1$, $\tan(\theta-\phi)$ is infinity, i.e. $\theta-\phi = \frac{1}{2}\pi$, and the two lines must therefore be perpendicular to each other.

This result, viz. that *when the product of the gradients of two straight lines is* -1 *they are at right angles*, is a most important one.

Here are some simple illustrations of the use of it.

(1) Are the lines whose equations are $3x-4y = 7$ and $4x+3y = 9$ perpendicular?

The gradient of the first is $\frac{3}{4}$ and of the second $-\frac{4}{3}$. The product of these is -1. $\therefore$ the lines are perpendicular.

(2) What condition must hold if the two lines $ax+by+c = 0$, $a'x+b'y+c' = 0$ are at right angles?

Their gradients are $-\dfrac{a}{b}$, $-\dfrac{a'}{b'}$ respectively.

$$\therefore \left(-\frac{a}{b}\right)\left(-\frac{a'}{b'}\right) = -1,$$

i.e. $$aa'+bb' = 0.$$

(3) The line $ax+by+c = 0$ will always be perpendicular to any one of the lines represented by $bx-ay+\lambda = 0$ for different values of λ.

This follows because the gradients are $-\dfrac{a}{b}$, $\dfrac{b}{a}$, respectively, whose product is -1.

(4) The line $\dfrac{x}{a}-\dfrac{y}{b}+\lambda = 0$ will always be perpendicular to the line $ax+by+c = 0$.

The reason is once again that the product of the gradients of the two lines is -1.

(5) To find the equation of a line through the point (h, k) which is perpendicular to the line whose equation is

$$ax+by+c = 0.$$

Any straight line through (h, k) has for its equation

$$y-k = m(x-h).$$

The gradient of this line is m, and if it is perpendicular to the line whose gradient is $-\frac{a}{b}$, we have at once $m\left(-\frac{a}{b}\right) = -1$, i.e. $m = \frac{b}{a}$.

$\therefore$ the required line is $y-k = \frac{b}{a}(x-h)$,

i.e. $$\mathbf{bx-ay = bh-ak}.$$

Thus a practical rule for quickly writing down the equation of a line through a given point perpendicular to a given straight line is:

(i) Interchange the coefficients of x and y and alter the sign of one of them.

(ii) Substitute the coordinates of the given point for x and y in the expression thus obtained.

(iii) Equate the results of (i) and (ii).

For example, to find the equation of a straight line through the point $(2, -3)$ perpendicular to the line whose equation is $4x-5y-9 = 0$.

Step 1. Write down $5x+4y$.

Step 2. Calculate $5(2)+4(-3) = -2$.

Step 3. The equation required is $5x+4y = -2$.

After a very little practice these three steps can be done mentally and the result obtained immediately.

The two straight lines whose equations are $ax+by+c = 0$, $a'x+b'y+c' = 0$ have gradients of $-\frac{a}{b}$, $-\frac{a'}{b'}$, respectively. Hence, with the previous notation,

$$\tan(\theta-\phi) = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi} = \frac{-\frac{a}{b}+\frac{a'}{b'}}{1+\frac{aa'}{bb'}} = \text{[illegible]}$$

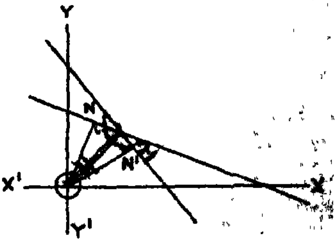
The lines are parallel if $\theta-\phi = 0$, $\tan(\theta-\phi) = 0$, and $\frac{a}{a'} = \frac{b}{b'}$.

The lines are perpendicular if $\theta-\phi = 90$, $\tan(\theta-\phi) = \infty$, and $\therefore\ aa'+bb' = 0$.

Suppose we have two straight lines whose equations are

$$x\cos\alpha+y\sin\alpha = p \quad \text{and} \quad x\cos\alpha'+y\sin\alpha' = p'.$$

Let ON, ON' be the perpendiculars from the origin to each of them respectively.



Then, with the usual notation,

$$\angle NOX = \alpha,\ \angle N'OX = \alpha'.$$

$$\therefore\ \angle NON' = \alpha-\alpha'.$$

But the angle between any two straight lines must be the same as that between two lines perpendicular to them.

$\therefore$ one of the angles between the two lines is $\alpha-\alpha'$.

The lines are clearly parallel if $\alpha = \alpha'$. And they are perpendicular if $\alpha\sim\alpha' = \frac{\pi}{2}$.

15. A homogeneous equation in x and y of the second degree represents two straight lines through the origin.

The most general equation of the second degree in x and y is

$$ax^2+2hxy+by^2 = 0.$$

Rewriting it $$\frac{y^2}{x^2}+\frac{2h}{b}\frac{y}{x}+\frac{a}{b} = 0,$$

we obtain a quadratic equation in $\frac{y}{x}$ which has two roots m_1, m_2 suppose.

Hence we have $$\left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right) = 0.$$

any point (x, y) on the locus represented by

$$ax^2+2hxy+by^2 = 0$$

must satisfy either $\frac{y}{x}-m_1 = 0$ or $\frac{y}{x}-m_2 = 0$, i.e. it must be either on the line whose equation is $y-m_1x = 0$ or on that whose equation is $y-m_2x = 0$; i.e. the locus is two straight lines each going through the origin.

16. To find the angle between the two straight lines represented by the equation $ax^2+2hxy+by^2 = 0$

As in the previous Article, let

$$\frac{y^2}{x^2}+\frac{2h}{b}\frac{y}{x}+\frac{a}{b} \equiv \left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right).$$

Then we have

$$m_1+m_2 = -\frac{2h}{b} \quad \text{and} \quad m_1m_2 = \frac{a}{b}.$$

Now the angle between the lines whose equations are $y-m_1x = 0$ and $y-m_2x = 0$ is $\tan^{-1}\frac{m_1-m_2}{1+m_1m_2}$.

But

$$(m_1-m_2)^2 = (m_1+m_2)^2-4m_1m_2 = \frac{4h^2}{b^2}-\frac{4a}{b} = \frac{4(h^2-ab)}{b^2}.$$

$$\therefore \tan^{-1}\frac{m_1-m_2}{1+m_1m_2} = \tan^{-1}\frac{2\sqrt{(h^2-ab)}}{a+b},$$

and this last expression is the acute angle between the two lines.

We note that if $h^2 = ab$, the angle is zero and the lines are coincident.

Also that if $a+b = 0$, the angle is $\tan^{-1}\infty = \frac{\pi}{2}$, i.e. the lines are perpendicular.

If $h^2 < ab$, the lines are imaginary.

17. To find the condition that the general equation of the second degree in x and y shall represent two straight lines

The most general equation of the second degree in x and y is

$$ax^2+2hxy+by^2+2gx+2fy+c = 0.$$

This is the standard form of such an equation, which should be memorized.

If the equation does represent two straight lines, then the left-hand side of it must have two real linear factors.

The condition for this is obtained by a purely algebraical process. First arrange the equation in descending powers of x, thus:

$$ax^2+2x(hy+g)+by^2+2fy+c = 0.$$

Now solve for x and we have

$$x = \frac{-(hy+g)\pm\sqrt{\{(hy+g)^2-a(by^2+2fy+c)\}}}{a}.$$

If x is real, the expression under the radical sign must be a perfect square, i.e.

$$y^2(h^2-ab)+2y(gh-af)+g^2-ac$$

is a perfect square.

$$\therefore\ (gh-af)^2 = (h^2-ab)(g^2-ac),$$

which reduces to

$$\mathbf{abc+2fgh-af^2-bg^2-ch^2 = 0.}$$

The left-hand side of this equation is called the *Discriminant* and usually denoted in this subject by Δ.

The student familiar with the determinant notation will observe that

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

18. Prove that if the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

represents two straight lines, they must be parallel to the lines through the origin whose equation is

$$ax^2+2hxy+by^2 = 0$$

When $\Delta = 0$,

$$y^2+\frac{2h}{b}xy+\frac{a}{b}x^2+\frac{2g}{b}x+\frac{2f}{b}y+\frac{c}{b}$$

must be identical with $(y-m_1x-c_1)(y-m_2x-c_2)$.

$$\therefore\ m_1+m_2 = -\frac{2h}{b}, \qquad m_1 m_2 = \frac{a}{b}.$$

But the two lines $y-m_1x-c_1 = 0$ and $y-m_2x-c_2 = 0$ are parallel to the lines $y-m_1x = 0$, $y-m_2x = 0$ which pass through the origin, i.e. they are parallel to the two lines whose equation is

$$(y-m_1x)(y-m_2x) = 0;$$

whence

$$y^2-(m_1+m_2)xy+m_1m_2x^2 = 0.$$

$$\therefore\ y^2+\frac{2h}{b}xy+\frac{a}{b}x^2 = 0,$$

i.e.

$$ax^2+2hxy+by^2 = 0,$$

which proves the theorem.

19. Position of a point relative to a given straight line

Let the equation of the given line be $ax+by+c = 0$.

The perpendicular from the origin $(0,0)$ on it is of length

$$\frac{c}{\sqrt{(a^2+b^2)}}.$$

The perpendiculars from any two points such as (x_1,y_1), (x_2,y_2) on it are respectively

$$\frac{ax_1+by_1+c}{\sqrt{(a^2+b^2)}} \quad\text{and}\quad \frac{ax_2+by_2+c}{\sqrt{(a^2+b^2)}}.$$

It follows that if ax_1+by_1+c and ax_2+by_2+c have each the same sign as c, the points (x_1,y_1), (x_2,y_2) are on the same side of the line as the origin. If either expression has a different sign to that of c, then the point to which the expression corresponds is on the opposite side of the line to the origin.

It is also clear, without any reference to the position of the origin, that if ax_1+by_1+c and ax_2+by_2+c have the same sign, the points (x_1,y_1), (x_2,y_2) are on the same side of the line $ax+by+c = 0$, whereas if they are of opposite signs, the points are on opposite sides of this line.

Consider, for example, the line $-3x-4y+5 = 0$, written with its absolute term positive.

Substitute the coordinates $(0,0)$, $(-2,3)$, $(-3,2)$ in the expression $-3x-4y+5$.

The results are 5, -1, 6, respectively.

This shows that the origin and the point $(-3,2)$ are on one side of the line, and the point $(-2,3)$ on the other side of it.

Again, suppose that we wish to determine how the two points $(4,1)$, $(3,-3)$ are situated with regard to a triangle ABC, the equations of whose sides BC, CA, AB are, respectively,

$$-2x-3y+7=0, \quad -5x-2y+23=0, \quad -3x+y+5=0,$$

each equation being arranged with the absolute term positive.

Solving the equations, we find that A, B, and C are respectively the points $(3,4)$, $(2,1)$, $(5,-1)$.

Substituting in turn $(3,4)$, $(4,1)$, and $(3,-3)$ in $-2x-3y+7$, the results are -11, -4, $+10$.

Hence $(4,1)$ is on the same side of BC as A, but $(3,-3)$ is on the opposite side of BC to A.

Substituting in turn $(2,1)$, $(4,1)$, $(3,-3)$ in $-5x-2y+23$, the results are 11, 1, 14, showing that all three points are on the same side of CA.

Now substituting $(5,-1)$, $(4,1)$, $(3,-3)$ in $-3x+y+5$, the results are -11, -6, -7, showing that all three points are on the same side of AB.

It is clear, therefore, that the point $(4,1)$ lies within the triangle ABC, and that the point $(3,-3)$ lies within the area formed by BC, and AB and AC produced.

All of which, it should be noted, has been deduced without any reference to the position of the origin.

20. To find the equation of the lines bisecting the angles between two given straight lines whose equations are $ax+by+c=0$, $a'x+b'y+c'=0$, these equations being so arranged that the absolute terms are positive

Suppose (X, Y) is any point on a bisector.

Then taking the denominators as positive, and since the perpendiculars from (X, Y) on the lines must be equal in magnitude, we have

$$\frac{aX+bY+c}{\sqrt{(a^2+b^2)}} = \pm\frac{a'X+b'Y+c'}{\sqrt{(a'^2+b'^2)}}.$$

If we take the upper sign, $aX+bY+c$ and $a'X+b'Y+c'$ must be either both positive or both negative.

But the origin lies on the positive side of each of the given lines.

$$\therefore\ \frac{ax+by+c}{\sqrt{(a^2+b^2)}} = +\frac{a'x+b'y+c'}{\sqrt{(a'^2+b'^2)}}$$

represents the bisector of the angle between the two straight lines in which the origin lies.

As an example, consider the lines

$$4x-3y+1 = 0, \qquad -24x-7y+19 = 0.$$

Then $$\frac{4x-3y+1}{\sqrt{(4^2+3^2)}} = \frac{-24x-7y+19}{\sqrt{(24^2+7^2)}}$$

is the bisector of the angle which contains the origin.

This equation reduces to

$$25(4x-3y+1) = 5(-24x-7y+19),$$

i.e. $$22x-4y-7 = 0.$$

21. To find the equation of the bisectors of the two straight lines through the origin represented by the equation

$$ax^2+2hxy+by^2 = 0$$

Suppose the lines $y^2+\frac{2h}{b}xy+\frac{a}{b}x^2 = 0$ are the same as the lines $(y-m_1x)(y-m_2x) = 0$.

Then $m_1+m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.

If (X, Y) be any point on a bisector, then

$$\frac{Y-m_1X}{\sqrt{(1+m_1^2)}} = \pm\frac{Y-m_2X}{\sqrt{(1+m_2^2)}}.$$

$$\therefore\ (Y-m_1X)^2(1+m_2^2) = (Y-m_2X)^2(1+m_1^2).$$

$$\therefore\ (X^2-Y^2)(m_1^2-m_2^2) = 2XY(m_1-m_2)(1-m_1m_2),$$

i.e.
$$\frac{X^2-Y^2}{1-m_1m_2} = \frac{2XY}{m_1+m_2}.$$

$$\therefore\ \frac{X^2-Y^2}{a-b} = \frac{XY}{h}.$$

$\therefore$ the point (X, Y) is always on one of the two straight lines through the origin whose equation is

$$\frac{\mathbf{x}^2-\mathbf{y}^2}{\mathbf{a}-\mathbf{b}} = \frac{\mathbf{xy}}{\mathbf{h}}.$$

The fact that the sum of the coefficients of x^2 and y^2 in this equation is zero shows that the two lines represented by it are at right angles. We have thus proved the geometrical proposition that the internal and external bisectors of an angle are perpendicular to each other.

22. To find the equation of the two straight lines joining the origin to the two points at which a given straight line cuts a curve represented by the general equation of the second degree in x and y

Let the equation of the given line be $lx+my = 1$.

The equation of the curve is

$$ax^2+2hxy+by^2+2gx+2fy+c = 0.$$

Make this last equation homogeneous by means of the equation $lx+my = 1$, thus:

$$ax^2+2hxy+by^2+2(gx+fy)(lx+my)+c(lx+my)^2 = 0. \quad (1)$$

Now apply the 'retrospective' method.

Equation (1) is homogeneous and of the second degree in x and y. $\therefore$ it represents two straight lines passing through the origin.

Further, if the coordinates of any point satisfy both of the equations

$$lx+my = 1,$$

$$ax^2+2hxy+by^2+2gx+2fy+c = 0,$$

it is clear these coordinates must also satisfy equation (1). But a point whose coordinates satisfy the equations of both the line and the curve is clearly a point common to both.

$\therefore$ equation (1) is that of the two straight lines through the origin joining it to the points of intersection of the straight line and the curve.

As a numerical illustration of the above, suppose we desire to find the equation of the two straight lines joining the origin to the points of intersection of the line whose equation is $3x-4y+5 = 0$ with the curve $3x^2+4xy+y^2+3x-5y+7 = 0$.

First write the equation of the line, $-\frac{3}{5}x+\frac{4}{5}y = 1$.

Using this to make the equation of the curve homogeneous, we have

$$3x^2+4xy+y^2+(3x-5y)\frac{4y-3x}{5}+7\frac{(4y-3x)^2}{25} = 0$$

as the required equation. And this reduces to

$$93x^2+67xy+37y^2 = 0.$$

Ex. 1. Find the equation of the bisectors of the angles between the two straight lines whose equation is

$$15y^2-14xy-8x^2+26x+26y-21 = 0,$$

and indicate which of them bisects the angle in which the origin lies.

$$15y^2-14xy-8x^2+26x+26y-21 = (3y-4x+7)(5y+2x-3).$$

The equations of the lines are accordingly

$$3y-4x+7 = 0 \quad \text{and} \quad -5y-2x+3 = 0,$$

and that of the bisector of the angle containing the origin is

$$\frac{3y-4x+7}{5} = \frac{-5y-2x+3}{\sqrt{29}},$$

which reduces to

$$x(4\sqrt{29}-10)-y(25+3\sqrt{29}) = 7\sqrt{29}-15.$$

The equation of the other bisector similarly reduces to

$$x(4\sqrt{29}+10)+y(25-3\sqrt{29}) = 7\sqrt{29}+15.$$

Ex. 2. Find the equation of two straight lines through the origin perpendicular to those whose equation is

$$15x^2-11xy-12y^2 = 0.$$

Suppose that

$$y^2+\tfrac{11}{12}xy-\tfrac{15}{12}x^2 \equiv (y-m_1x)(y-m_2x).$$

Then $\qquad m_1+m_2 = -\tfrac{11}{12}, \qquad m_1m_2 = -\tfrac{5}{4}.$

We require the lines whose equation is

$$\left(y+\frac{1}{m_1}x\right)\left(y+\frac{1}{m_2}x\right) = 0,$$

i.e. $$y^2+\frac{m_1+m_2}{m_1m_2}xy+\frac{1}{m_1m_2}x^2 = 0,$$

i.e. $$15y^2+11xy-12x^2 = 0.$$

Ex. 3. Determine the single equation representing the internal and external bisectors of the angles between the straight lines $2x+y+3 = 0$, $x+y+2 = 0$. [W.]

The equations of the bisectors are

$$\frac{2x+y+3}{\sqrt{5}} = \pm\frac{x+y+2}{\sqrt{2}},$$

or as one equation

$$2(2x+y+3)^2 = 5(x+y+2)^2;$$

whence $$3x^2-2xy-3y^2+4x-8y-2 = 0.$$

Ex. 4. Find the value of K such that the lines joining the origin to the points of intersection of the line $2x-5y-K = 0$ with the conic whose equation is $3x^2+8xy-9y^2+5x-4y = 0$ may be perpendicular.

Rewrite the equation of the line $\dfrac{2}{K}x-\dfrac{5}{K}y = 1.$

Make the equation of the conic homogeneous by means of this result and we have

$$3x^2+8xy-9y^2+\frac{(2x-5y)}{K}(5x-4y) = 0.$$

The two straight lines through the origin represented by this equation are at right angles if the sum of the coefficients of x^2 and y^2 is zero, i.e. if

$$3+\frac{10}{K}-9+\frac{20}{K}=0;$$

whence $$K=5.$$

Ex. 5. If the equation $4x^2-2xy+3y^2-8x-6y+4+\lambda=0$ represents two straight lines, what is the value of λ?

Using the discriminant formula, we have

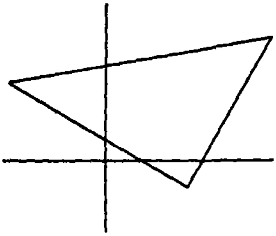
$$4.3(4+\lambda)+2(-3)(-4)(-1)-4(-3)^2-3(-4)^2-(4+\lambda)(-1)^2$$

equal to zero.

$$\therefore\ 48+12\lambda-24-36-48-4-\lambda=0.$$

$$\therefore\ \lambda=\tfrac{64}{11}.$$

Ex. 6. Find the coordinates of the centre of the inscribed circle of the triangle whose vertices are the points $(4,3)$, $(-2,2)$, $(2,-1)$. The equations of the three sides of the triangle are



$$x-6y+14=0,\quad 2x-y-5=0,$$

and $$3x+4y-2=0.$$

The bisector of the angle between the lines $x-6y+14=0$ and $-3x-4y+2=0$ which does not contain the origin is

$$\frac{x-6y+14}{\sqrt{37}}=-\frac{-3x-4y+2}{5}=\frac{3x+4y-2}{5}.$$

Similarly, the bisector of the angle between $-2x+y+5=0$ and $-3x-4y+2=0$ which does not contain the origin is

$$\frac{-2x+y+5}{\sqrt{5}}=-\frac{-3x-4y+2}{5}=\frac{3x+4y-2}{5}.$$

Taking $\sqrt{37}=6{\cdot}1$ and $\sqrt{5}=2{\cdot}2$, these equations become

$$13{\cdot}3x+54{\cdot}4y=82{\cdot}2,\qquad 7{\cdot}4x-1{\cdot}8y=13,$$

of which the solution is approximately $x=2$, $y=1$.

EXAMPLES 5

1. Find the angle between the straight lines whose equations are

(1) $y-3x = 4,\ y+5x = 6.$

(2) $3x+4y+5 = 0,\ 5x-12y+3 = 0.$

(3) $28x^2-5xy-3y^2 = 0.$

(4) $12x^2-26xy+10y^2+11x-9y+2 = 0.$

2. Remembering that $bx-ay = bh-ak$ is the equation of the straight line through the point (h, k) perpendicular to the line $ax+by+c = 0$, write down by inspection the equations of the lines through the point $(-5, -3)$ which are respectively perpendicular to the lines

(1) $4x-7y+3 = 0$, (2) $x\cos\alpha+y\sin\alpha = 5$, (3) $\dfrac{x}{3}+\dfrac{y}{4} = 1.$

3. If the equation $5x^2-8xy+ay^2 = 0$ represents two perpendicular lines, what is the value of a?

4. Find the equations of the lines

(1) through the point $(1, -1)$ making an angle of 45° with the line $2x-y = 7$;

(2) through the point $(3, 2)$ making an angle of $\tan^{-1}\frac{3}{4}$ with the line $x-2y = 1$;

(3) through the point $(2, 1)$ parallel to $2x+3y = 5$.

5. Find the angle between the lines $x^2+2\sec\theta\, xy+y^2 = 0$.

6. Show that the following equations represent two straight lines and find the angle between them:

(1) $xy = 0.$

(2) $xy+4x-3y-12 = 0.$

(3) $x^2-y^2+4x+4 = 0.$

(4) $6x^2-19xy+3y^2+33x-14y+15 = 0.$

(5) $\sqrt{3}x^2+xy(1+\sqrt{3})+y^2-x(6+5\sqrt{3})-11y+30 = 0.$

7. For what values of λ do the following equations represent two straight lines?

(1) $\lambda xy+20x-21y-28 = 0.$ ✓

(2) $2x^2+5xy-3y^2-\lambda x+16y-21 = 0.$

(3) $12x^2-\lambda xy-x+28y-20 = 0.$

8. Find the equations of the bisectors of the angles between the straight lines whose equations are

(1) $y = 2x$, $y = 3x$.

(2) $4x-3y+5 = 0$, $5x+12y+6 = 0$.

(3) $3x-4y = 7$, $24x+7y-6 = 0$.

9. One bisector of the angle between the lines $4x-3y = 7$ and $5x-12y = 8$ is $9x+7y = 17$; what is the other? Which is the bisector of the angle within which the origin lies?

10. Find the equation of the bisectors of the angles between the lines represented by

(1) $(x^2+y^2)\cos\alpha+2xy\tan\alpha = 0$.

(2) $4x^2-8xy+3y^2 = 0$.

(3) $a(x+y)^2+b(x-y)^2+c(x+y)(x-y) = 0$.

11. Find the coordinates of the centre of the inscribed circle of the triangles formed by

(1) the lines $x = 0$, $y = 0$, $3x+4y = 5$;

(2) the lines $4x+3y = 5$, $7x+24y+25 = 0$, $5x-12y+13 = 0$;

(3) joining the three points whose coordinates are $(0, 4)$, $(2, 5)$, $(-3, 4)$.

12. Find the coordinates of the centre of the circumcircle of the triangle whose vertices are the points $(4, 3)$, $(-2, 2)$, $(2, 1)$ from the consideration that it is the point of intersection of the perpendicular bisectors of two of the sides of the triangle.

13. Show that the angles between the straight lines represented by $3x^2-6xy+y^2 = 0$ and $x^2-\sqrt{10}xy+y^2 = 0$ are equal.

14. Find the equation of the two straight lines joining the origin to the points of intersection of the line $2x+3y = 1$ with the curves

(1) $x^2+y^2 = 9$.

(2) $y^2 = 4ax$.

(3) $4(x-3)^2+9(y-2)^2 = 1$.

15. Find the coordinates of the in-centre of the triangle formed by the lines

$$(t^2-1)x+2ty = 1, \quad (u^2-1)x+2uy = 1, \quad (v^2-1)x+2vy = 1.$$

16. Find the equation of the lines joining the origin to the common points of $y = mx+c$ and $x^2+y^2+2gx = 0$. What relation must there be between the constants c, m, and g if the lines are at right angles to each other?

17. Find the coordinates of the image of the point (2, 3) in the line $3x-4y = 10$.

18. Find the coordinates of the orthocentre of the triangles formed by the lines

(1) $3x-2y+1 = 0$, $x+y = 3$, $x-3y+9 = 0$.

(2) $tx+y = t^2+1$, $ux+y = u^2+1$, $vx+y = v^2+1$.

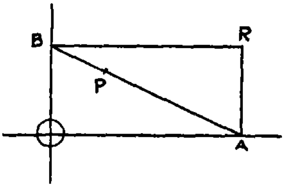
19. Show that the equation $y^3-3x^2y+3xy^2-x^3 = 0$ represents three straight lines through the origin equally inclined to one another.

20. If $ab = h^2$ and $af^2 = bg^2$, show that the general equation of the second degree $ax^2+2hxy+by^2+2gx+2fy+c = 0$ represents two parallel straight lines.

23. Typical examples on straight lines

1. Through a fixed point P (a, b) a straight line is drawn cutting rectangular axes in A and B. The rectangle $OARB$ is then completed. Find the locus of R.

Let R be the point (X, Y).



Then, as in the diagram, $X = OA$, $Y = OB$.

$\therefore$ the equation of AB is $\frac{x}{X}+\frac{y}{Y} = 1$.

But this goes through P the point (a, b).

$$\therefore \frac{a}{X}+\frac{b}{Y} = 1,$$

which is the equation of the required locus of R.

2. A straight line moves so that the algebraic sum of the perpendicular distances from it of a number of fixed points is always zero. Show that it must always pass through a fixed point.

Suppose that the points are (a_1, b_1), (a_2, b_2),..., (a_n, b_n).

Then the lengths of the perpendiculars from these points on a line whose equation is $x\cos\alpha+y\sin\alpha = p$ are

$$a_1\cos\alpha+b_1\sin\alpha-p, \quad \text{etc.}$$

$\therefore$ by data,

$$(a_1+a_2+\ldots+a_n)\cos\alpha+(b_1+b_2+\ldots+b_n)\sin\alpha = np,$$

which shows that the line $x\cos\alpha+y\sin\alpha = p$ must always pass through the fixed point whose coordinates are

$$\frac{a_1+a_2+\ldots+a_n}{n}, \qquad \frac{b_1+b_2+\ldots+b_n}{n},$$

i.e. the centre of mean position of the n fixed points.

3. Prove that the product of the perpendiculars from a point (f, g) on the lines whose equation is $ax^2+2hxy+by^2 = 0$ is

$$\frac{af^2+2fgh+bg^2}{\sqrt{\{(a-b)^2+4h^2\}}}.$$

Let $$y^2+\frac{2h}{b}xy+\frac{a}{b}x^2 \equiv (y-m_1x)(y-m_2x).$$

The product required is

$$\frac{(g-m_1f)(g-m_2f)}{\sqrt{\{(1+m_1^2)(1+m_2^2)\}}}.$$

Now $$m_1+m_2 = -\frac{2h}{b}, \qquad m_1m_2 = \frac{a}{b}.$$

$$\therefore\ m_1^2+m_2^2 = (m_1+m_2)^2-2m_1m_2 = \frac{4h^2}{b^2}-\frac{2a}{b}.$$

$$\therefore\ (1+m_1^2)(1+m_2^2) = 1+\frac{4h^2}{b^2}-\frac{2a}{b}+\frac{a^2}{b^2} = \frac{(a-b)^2+4h^2}{b^2}$$

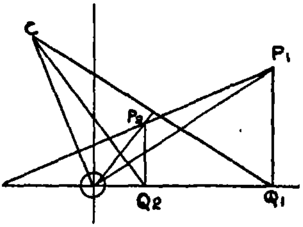
and $$g^2-(m_1+m_2)fg+m_1m_2f^2 = \frac{af^2+2fgh+bg^2}{b}.$$

$\therefore$ the product required is

$$\frac{af^2+2fgh+bg^2}{\sqrt{\{(a-b)^2+4h^2\}}}.$$

4. P_1Q_1, P_2Q_2 are any two lines perpendicular to a third line OQ_1Q_2. Prove that the perpendicular from Q_2 on to OP_1 and the perpendicular from Q_1 on to OP_2 meet on the perpendicular from O on to P_1P_2.

Take OQ_1Q_2 as x-axis and a line through O perpendicular to it as y-axis.



Let P_1 be the point $(\alpha, m_1\alpha)$, P_2 $(\beta, m_2\beta)$.

The equation of a line through the point Q_1 $(\alpha, 0)$ perpendicular to $y-m_2x = 0$ is

$$x+m_2y = \alpha.$$

Similarly, the equation of a line through the point Q_2 $(\beta, 0)$ perpendicular to $y-m_1x = 0$ is

$$x+m_1y = \beta.$$

The point C where these two lines intersect has therefore as its coordinates

$$\frac{m_1\alpha-m_2\beta}{m_1-m_2} \quad\text{and}\quad \frac{\beta-\alpha}{m_1-m_2},$$

and the equation of OC is

$$\frac{x}{m_1\alpha-m_2\beta} = \frac{y}{\beta-\alpha}.$$

The gradient of the line P_1P_2 is clearly

$$\frac{m_1\alpha-m_2\beta}{\alpha-\beta}.$$

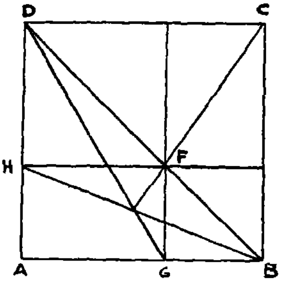
$\therefore$ the product of the gradients of OC and P_1P_2 is -1.

$\therefore$ OC is perpendicular to P_1P_2, i.e. C lies on a line through O perpendicular to P_1P_2.

5. $ABCD$ is a square, and through any point F on BD parallels are drawn to the sides cutting AB at G, and AD at H. Prove that CF, BH, DG are concurrent.

Take AB, AD as axes of coordinates.

Let F be the point (α, β), and let the side of the square be of length a. Then the equation of BH is



$$\frac{x}{AB}+\frac{y}{AH} = 1,$$

i.e.
$$\frac{x}{a}+\frac{y}{\beta} = 1.$$

Similarly, the equation of DG is

$$\frac{x}{\alpha}+\frac{y}{a} = 1.$$

The equation of a line through the intersection of BH and DG is

$$\beta x+ay-a\beta-\lambda(ax+\alpha y-a\alpha) = 0.$$

If this passes through the point (α, β), we have, by substitution, $\lambda = 1$.

The line joining F to the intersection of the lines BH, DG has therefore as its equation

$$(\beta-a)x+(a-\alpha)y+a(\alpha-\beta) = 0.$$

But $x = a$, $y = a$ clearly satisfies this equation.

$\therefore$ C lies on the line.

$\therefore$ CF, BH, DG are concurrent.

6. The sides of a parallelogram are parallel to $x-y = 0$, $2x+y = 0$. The extremities of a diagonal are the points $(1, 1)$, $(-2, 3)$. Find the equation of the other diagonal.

The middle point of the diagonal is $(-\frac{1}{2}, 2)$. Let the equation of the other diagonal be $y-2 = m(x+\frac{1}{2})$.

A line through $(1, 1)$ parallel to $y = x$ is $y-1 = x-1$, i.e. $y = x$. A line through $(-2, 3)$ parallel to $2x+y = 0$ is $2(x+2)+y-3 = 0$. The last pair intersect at the point $x = y = -\frac{1}{3}$.

As this point is on the diagonal $2y-4 = m(2x+1)$, we have $m = -14$, and the required equation is $14x+y+5 = 0$.

7. Find the equation of a line which makes the same angle with the line $ax+by+c = 0$ as the line $px+qy+r = 0$ does and at the same point in the line.

The equation of any line through the intersection of the two given lines is

$$ax+by+c+\lambda(px+qy+r) = 0. \qquad (1)$$

The angle between $ax+by+c = 0$ and $px+qy+r = 0$ is

$$\tan^{-1}\frac{-\frac{a}{b}+\frac{p}{q}}{1+\frac{ap}{bq}} = \tan^{-1}\frac{bp-aq}{ap+bq}.$$

The angle between (1) and $ax+by+c = 0$ is

$$\tan^{-1}\frac{-\frac{a}{b}+\frac{a+\lambda p}{b+\lambda q}}{1+\frac{a(a+\lambda p)}{b(b+\lambda q)}} = \tan^{-1}\frac{\lambda(bp-aq)}{a^2+b^2+\lambda(ap+bq)}.$$

$\therefore$ by data $$\frac{aq-bp}{ap+bq} = \frac{\lambda(bp-aq)}{a^2+b^2+\lambda(ap+bq)};$$

whence $$\lambda = -\frac{a^2+b^2}{2(ap+bq)},$$

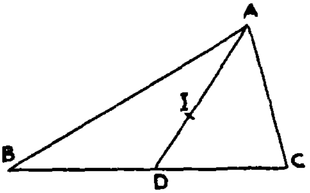
and (1) becomes

$$2(ax+by+c)(ap+bq) = (a^2+b^2)(px+qy+r).$$

8. Prove that the centre of the inscribed circle of a triangle ABC whose vertices are the points (x_1, y_1), (x_2, y_2), (x_3, y_3) is the point whose coordinates are

$$\frac{ax_1+bx_2+cx_3}{a+b+c}, \quad \frac{ay_1+by_2+cy_3}{a+b+c},$$

a, b, c being the sides of the triangle.



Let AD bisect the angle BAC and cut BC in D.

Then we know that

$$BD : DC = BA : AC = c : b.$$

$\therefore$ D is the point whose coordinates are

$$\frac{bx_2+cx_3}{b+c}, \quad \frac{by_2+cy_3}{b+c}.$$

Again, $ID : AD =$ in-radius : altitude $= \dfrac{\Delta}{s} : \dfrac{2\Delta}{a}$

$$= a : a+b+c.$$

$$\therefore\ AI : ID = b+c : a.$$

$\therefore$ the abscissa of the point I is

$$ax_1+(b+c)\frac{bx_2+cx_3}{b+c}, \text{ divided by } a+b+c,$$

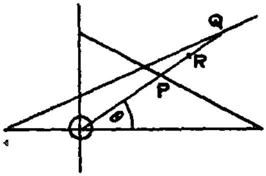
i.e.
$$\frac{ax_1+bx_2+cx_3}{a+b+c}.$$

Similarly, the ordinate of I is

$$\frac{ay_1+by_2+cy_3}{a+b+c}.$$

9. If a variable straight line is drawn through a fixed point O to cut two given straight lines in P and Q, to find the locus of R if OR be the harmonic mean of OP and OQ.

Take the point O as origin and any two fixed lines at right angles through it as axes of coordinates.



Let the two given lines have as their equations

$$ax+by+c = 0, \quad a'x+b'y+c' = 0.$$

Then, by data,

$$\frac{2}{OR} = \frac{1}{OP}+\frac{1}{OQ}.$$

Let OPQ make an angle of θ with the x-axis.

Then
$$a.OP\cos\theta+b.OP\sin\theta+c = 0.$$

$$\therefore\ \frac{1}{OP} = -\frac{a\cos\theta+b\sin\theta}{c}.$$

Similarly,
$$\frac{1}{OQ} = -\frac{a'\cos\theta+b'\sin\theta}{c'}.$$

$$\therefore\ \frac{2}{OR} = -\left(\frac{a}{c}+\frac{a'}{c'}\right)\cos\theta-\left(\frac{b}{c}+\frac{b'}{c'}\right)\sin\theta.$$

$$\therefore\ 2cc'+(ac'+a'c)OR\cos\theta+(bc'+b'c)OR\sin\theta = 0,$$

or, if (x, y) be the cartesian coordinates of R,

$$(ac'+a'c)x+(bc'+b'c)y+2cc' = 0,$$

which is the equation of a fixed straight line.

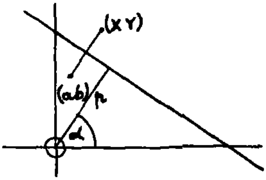
10. Prove that the coordinates of the reflection of the point (a, b) in the line

$$x\cos\alpha+y\sin\alpha-p = 0$$

are $2p\cos\alpha-a\cos2\alpha-b\sin2\alpha$

and $2p\sin\alpha-a\sin2\alpha+b\cos2\alpha$.

[O. and C.]



Let (X, Y) be the required point which is the reflection of the point (a, b) in the given straight line.

Then

$$X\cos\alpha+Y\sin\alpha-p = -(a\cos\alpha+b\sin\alpha-p).$$

$$\therefore\ X\cos\alpha+Y\sin\alpha+a\cos\alpha+b\sin\alpha-2p = 0.$$

Again,
$$\frac{Y-b}{X-a} = \tan\alpha.$$

$$\therefore\ X\sin\alpha-Y\cos\alpha-a\sin\alpha+b\cos\alpha = 0;$$

whence

$$X(\cos^2\alpha+\sin^2\alpha)+a(\cos^2\alpha-\sin^2\alpha)+b\sin2\alpha-2p\cos\alpha = 0,$$

i.e.
$$X+a\cos2\alpha+b\sin2\alpha = 2p\cos\alpha.$$

Similarly,
$$Y+a\sin2\alpha-b\cos2\alpha = 2p\sin\alpha.$$

From which the values of X and Y at once follow.

11. Prove that the equation $x^2-xy-3y^2-3x-5y-1 = 0$ represents two straight lines, and find the coordinates of the point of intersection.

Find also the equation which represents the two straight lines which may be drawn through their point of intersection perpendicular to them; and show that, if these lines cut the axis Ox in X, X' and the axis Oy in Y, Y', the equation of the line joining the mid-points of XX' and YY' is

$$6x-14y = 7. \qquad \text{[C.]}$$

Write the equation

$$x^2-x(y+3)-(3y^2+5y+1) = 0;$$

whence

$$x = \frac{y+3\pm\sqrt{(13y^2+26y+13)}}{2} = \frac{y+3\pm\sqrt{13}(y+1)}{2};$$

whence the equations of the two lines are

$$2x-y(1+\sqrt{13})-3-\sqrt{13} = 0$$

and

$$2x-y(1-\sqrt{13})-3+\sqrt{13} = 0.$$

Subtracting, $y = -1$, $x = 1$.

The lines through $(1, -1)$ perpendicular to the above are

$$(x-1)(1+\sqrt{13})+2(y+1) = 0$$

and

$$(x-1)(1-\sqrt{13})+2(y+1) = 0,$$

or, as one equation,

$$-12(x-1)^2+4(y+1)^2+4(x-1)(y+1) = 0,$$

i.e.

$$3x^2-xy-y^2-7x-y+3 = 0.$$

Where $x = 0$, $\quad y^2+y-3 = 0$, $\quad \therefore\ y_1+y_2 = -1$;

$y = 0$, $\quad 3x^2-7x+3 = 0$, $\quad \therefore\ x_1+x_2 = \frac{7}{3}$,

where OX_1 and $OX_2 = x_1$ and x_2 and OY_1 and $OY_2 = y_1$ and y_2 respectively.

$\therefore$ the coordinates of the mid-points of X_1X_2 and Y_1Y_2 are $(\frac{7}{6}, 0)$ and $(0, -\frac{1}{2})$, and the equation of the line joining them is $\frac{6x}{7}-2y = 1$, i.e. $6x-14y = 7$.

12. Prove that the condition that two of the three straight lines

$$ax^3+bx^2y+cxy^2+dy^3 = 0$$

are at right angles is $a^2+d^2+ac+bd = 0$.

Suppose that m_1, m_2, m_3 are the three roots of the equation

$$\left(\frac{y}{x}\right)^3+\frac{b}{d}\left(\frac{y}{x}\right)^2+\frac{c}{d}\left(\frac{y}{x}\right)+\frac{a}{d}=0.$$

The data inform us that $m_1 m_2 = -1$.

Hence $$-\frac{a}{d}=m_1 m_2 m_3=-m_3.$$

$\therefore$ $\frac{a}{d}$ satisfies the equation in $\frac{y}{x}$, i.e.

$$\frac{a^3}{d^3}+\frac{ca^2}{d^3}+\frac{ba}{d^2}+\frac{a}{d}=0.$$

$$\therefore\ a^2+ac+bd+d^2=0.$$

13. Find the equation of the straight line drawn through the point (x',y') perpendicular to the straight line $lx+my+n=0$.

Prove that the coordinates of the reflection of this point in this straight line are given by the equations

$$\frac{x-x'}{l}=\frac{y-y'}{m}=-\frac{2(lx'+my'+n)}{l^2+m^2}.$$ [O. and C.]

The equation of the straight line through (x',y') perpendicular to the line $lx+my+n=0$ can be written down at once as

$$m(x-x')=l(y-y'). \tag{1}$$

Let (X,Y) be the reflection of (x',y') in the line

$$lx+my+n=0.$$

Then $$\frac{lX+mY+n}{\sqrt{(l^2+m^2)}}=-\frac{lx'+my'+n}{\sqrt{(l^2+m^2)}}.$$

If (X,Y) is on the line (1), then $m(X-x')=l(Y-y')$.

$$\therefore\ \frac{X-x'}{l}=\frac{Y-y'}{m}=\frac{lX-lx'+mY-my'}{l^2+m^2}$$

$$=\frac{-2(lx'+my'+n)}{l^2+m^2}.$$

EXAMPLES 6

1. If u and v are variable quantities connected by the relation $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$, show that the line joining the points $(u, 0)$, $(0, v)$ passes through the point (f, f).

2. Find the equation of the line drawn through the point of intersection of the lines $10x-4y-43=0$ and $5x+3y-11=0$, which is perpendicular to the line $5x-4y+3=0$. [L.]

3. Prove that as the value of t varies the point $x=at+2$, $y=2t+b$ moves along a straight line, and find the values of a, b in order that this line may be the line $2x-3y+5=0$.

Find the coordinates of the two points on this last line at a distance $\sqrt{2}$ from the origin. [W.]

4. Prove that the equation

$$x^2+4xy+3y^2-12x-20y+32=0$$

represents two straight lines, and that the intercept which they make on the line $x+2y=5$ subtends a right angle at the origin. [D.]

5. The equation $12x^2+7xy+ky^2-31x+17y+7=0$ represents two straight lines; find the value of k and the angle between the lines. [D.]

6. Find the equation of two straight lines which are parallel to $x\cos\alpha+y\sin\alpha=p$, and are at a distance of d from it on opposite sides.

7. The sides of a parallelogram are parallel to $5x-y=0$ and $x+3y=0$, and the points $(4, 3)$, $(-6, -1)$ are the extremities of a diagonal. Find the equation of the other diagonal.

8. Show that for any value of m

$$m(x-3y)+3m-x+y=0$$

passes through a fixed point, and find its coordinates.

9. Show that the lines $ax^2+2hxy+by^2=k(x^2+y^2)$ and $ax^2+2hxy+by^2=0$ have the same bisectors.

10. Prove that the equation

$$x^2+6xy+5y^2+x+17y-12=0$$

represents two straight lines, and find their point of intersection.

11. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, show that the equation of the line joining their point of intersection to the origin is

$$y\surd(bc-f^2) = x\surd(ac-g^2).$$

12. Find an equation for λ, giving values other than zero, for which the equation

$$\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+(x-h)^2+(y-k)^2 = 0$$

is that of a pair of straight lines. [O. and C.]

13. $ABCD$ is a parallelogram, the coordinates of A, B, C being $(5,0)$, $(-1,2)$, $(\frac{4}{5},\frac{28}{5})$, respectively. If E and F are the feet of the perpendiculars from the origin on AC and BC, prove that D, E, F are collinear. [W.]

14. $OAPB$ is a parallelogram of which the side OA lies along the x-axis, OB along the line $y = 2x$, and P is the point (X,Y). Through A and B lines are drawn perpendicular to OA and OB meeting at Q. Find the coordinates of Q in terms of X and Y. Also find the locus of Q when P describes the line $x+2y = 1$. [B.]

15. Prove that if $l:m$ is the ratio in which the straight line $ax+by+c=0$ divides the line joining (x_1,y_1) and (x_2,y_2), then

$$\frac{l}{m} = -\frac{ax_1+by_1+c}{ax_2+by_2+c}.$$

16. Find the equation of the straight lines through the point $(2,\frac{4}{3})$ which will divide the triangle whose vertices are $(0,0)$, $(\frac{15}{4},\frac{3}{2})$, $(\frac{4}{3},\frac{8}{3})$ into two parts which are of equal area. [O.]

17. Show that the locus of a point whose distances from two fixed straight lines $x\cos\alpha+y\sin\alpha = p$ and $x\cos\beta+y\sin\beta = q$ are in a constant ratio, is a line passing through the point of intersection of the two given lines.

18. Find the equation of the straight lines drawn through the intersection of the lines $lx+my = 1$, $l'x+m'y = 1$, which are (1) parallel to, (2) perpendicular to, the line $ax+by = 1$.

19. Show that the foot of the perpendicular from the point $(a,0)$ to the line $x-ty+at^2 = 0$, where t varies, is always on the y-axis.

20. Find the area of the triangle formed by the lines

$$3x^2-4xy+y^2 = 0 \quad \text{and} \quad 2x-y = 6.$$

21. Lines through the points $(-a, 0)$, $(a, 0)$ make angles of θ and $\theta+\alpha$ with the x-axis respectively. Show that the locus of their points of intersection for varying values of θ is

$$x^2+y^2-2ay\cot\alpha = a^2.$$

22. Perpendicular straight lines are drawn through the fixed point $(0, a)$ to meet the x-axis at A and B. An equilateral triangle ABC is described with AB as base, on either side of the x-axis. Prove that the equation of the locus of C is $y^2 = 3(x^2+a^2)$. [W.]

23. In a triangle ABC, AO is the perpendicular to the base. If BOC and AO be taken as axes of coordinates, find the coordinates of (1) the orthocentre, (2) the centroid, (3) the circumcentre of triangle ABC, and show that all three are collinear.

24. Find the equations of the lines through a point (x', y') perpendicular to the lines $ax^2+2hxy+by^2 = 0$.

25. A parallelogram is formed by the lines $ax^2+2hxy+by^2 = 0$ and those through (x', y') parallel to them, Prove that the equation of the diagonal which does not pass through the origin is

$$(2x-x')(ax'+hy')+(2y-y')(hx'+by') = 0. \quad \text{[O. and C.]}$$

26. Show that the area of the parallelogram in the previous example is

$$\frac{ax'^2+2hx'y'+by'^2}{2\sqrt{(h^2-ab)}}.$$

27. Find the equation of the locus of the point the product of the perpendiculars from which to the lines $2x^2+4xy-y^2 = 0$ is equal to the product of the perpendiculars from it to the lines $x^2-4y^2 = 0$.

28. Prove that the equation $x^3+6x^2y-3xy^2-2y^3 = 0$ represents three straight lines through the origin which make equal angles with each other.

29. Find the coordinates of the reflection of the point $(4, 3)$ in the line $5x-12y = 153$.

30. The distance of a point $(3, 2)$ from each of two straight lines which pass through the origin is 5; find their equation.

31. A straight line PAB is drawn through a point P to meet two perpendicular straight lines in A and B. A point C is taken on AB such that

$$\frac{m+n}{PC} = \frac{m}{PA} + \frac{n}{PB}.$$

Show that the locus of C is a straight line passing through the intersection of the two given lines.

32. Find the polar equation of the line through the point (r_1, θ_1) perpendicular to the line whose equation is $r\cos(\theta-\alpha) = p$.

Formulae of Chapter II for reference

1. Forms of the equation of the straight line are

$$Ax+By+C = 0, \qquad y = mx+c,$$

$$\frac{x}{a}+\frac{y}{b} = 1, \qquad lx+my = 1,$$

$$x\cos\alpha+y\sin\alpha = p.$$

2. The equation of the line joining two points (x_1, y_1) (x_2, y_2) is

$$y-y_1 = m(x-x_1),$$

where

$$m = \frac{y_1-y_2}{x_1-x_2}.$$

3. The length of the perpendicular from (h, k) on $ax+by+c = 0$ is

$$\frac{ah+bk+c}{\sqrt{(a^2+b^2)}}.$$

4. The area of a triangle whose vertices are the points (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\tfrac{1}{2}(x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3).$$

5. The angle between the lines $y = m_1x$, $y = m_2x$ is

$$\tan^{-1}\frac{m_1-m_2}{1+m_1m_2}.$$

6. If the lines $y-m_1x = 0$, $y-m_2x = 0$ are perpendicular,

$$m_1m_2 = -1.$$

7. The angle between the two lines $ax^2+2hxy+by^2 = 0$ is

$$\tan^{-1}\frac{2\sqrt{(h^2-ab)}}{a+b}.$$

8. The bisector of the two lines

$$a_1x+b_1y+c_1 = 0, \qquad a_2x+b_2y+c_2 = 0$$

in which the origin lies is, if c_1 and c_2 are positive,

$$\frac{\mathbf{a_1x+b_1y+c_1}}{\sqrt{(\mathbf{a_1^2+b_1^2})}} = \frac{\mathbf{a_2x+b_2y+c_2}}{\sqrt{(\mathbf{a_2^2+b_2^2})}}.$$

9. The equation of the bisectors of the lines

$$ax^2+2hxy+by^2 = 0$$

is

$$\frac{\mathbf{x^2-y^2}}{\mathbf{a-b}} = \frac{\mathbf{xy}}{\mathbf{h}}.$$

10. The general equation $ax^2+2hxy+by^2+2gx+2fy+c = 0$ represents two straight lines if the discriminant

$$\mathbf{abc+2fgh-af^2-bg^2-ch^2 = 0}.$$

III

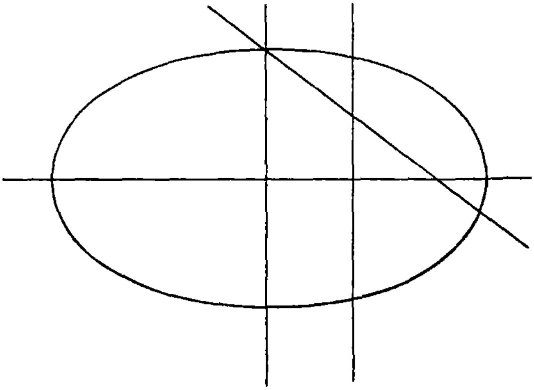
EQUATIONS. TRANSFERENCE OF AXES. TANGENTS

ANY equation involving two variable quantities x and y is really a statement of a relation existing between them. In other words, x and y, when connected by an equation, are not independent of each other, for when one is known the other can be determined and has a definite value or values.

Take the case of a linear equation such as $3x+4y = 12$. If $x = 2$, it follows that y must be $1\frac{1}{2}$, and that this is the only value of y corresponding to the value 2 of x.

Again, take an equation of the second degree in x and y such as $9x^2+25y^2 = 225$. If $x = 2$, y has the two values $\pm\frac{1}{5}\sqrt{189} = \pm 2{\cdot}75$ approximately, and these are the only values y can have when $x = 2$.

The diagram below represents the curve $9x^2+25y^2 = 225$ and the two straight lines $x = 2$ and $3x+4y = 12$.



It will be observed from it that there is only one point where the line $x = 2$ cuts the line $3x+4y = 12$, but two points where each of the lines cuts the curve.

In geometry the locus of a point is defined as the path which it must trace out when moving under some geometrical restriction. In analytical geometry we take the coordinates of the moving point as (x, y) and express the geometrical restriction in the form of an equation in x and y. When required to find the locus of a point, this equation can be regarded as the answer, which it is not essential to interpret any further, unless specifically asked to do so.

We know that the square of the distance of any point (x, y) from the origin is x^2+y^2.

Suppose that the locus of the point was required, subject to the condition that it had always to be at a fixed distance a from the origin.

Then clearly $$x^2+y^2 = a^2,$$ and this is the locus of the point which satisfies the stipulated condition.

Consider again the locus of a point which moves so that the square of its distance from one fixed line is always five times the cube of its distance from a line at right angles to the former line.

Take the fixed lines as axes of coordinates.

Let P be a point (x, y) on the locus. Draw PM, PN perpendicular to the axes.

The geometrical restriction here imposed upon the movement of P is expressed geometrically by saying that

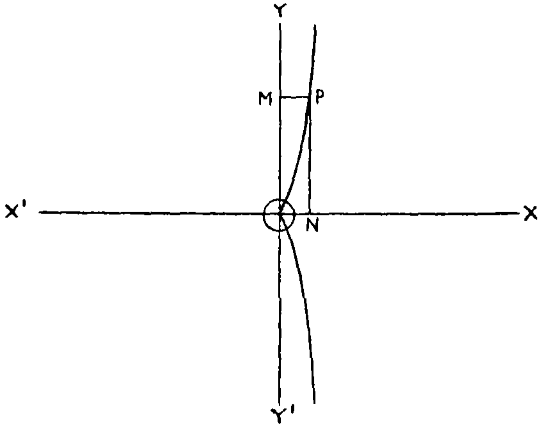
$$PN^2 = 5PM^3.$$

But since $PM = x$ and $PN = y$, we can say analytically that $y^2 = 5x^3$ is the equation of the locus of P.

We will now proceed to examine this equation $y^2 = 5x^3$ and see what facts can be gleaned from it.

Firstly, we see that $x = 0$, $y = 0$ satisfies it. This means that the curve represented by the equation passes through the origin. Conversely, if we had any equation in x and y to deal with which $x = 0$, $y = 0$ did not satisfy, we should know that the curve it represented did not pass through the origin. Next we observe that for any negative value of x, y^2 becomes equal to a negative quantity, i.e. y is imaginary. This means that no point on the curve can lie in the second or third quadrant where x would be negative. Hence we infer that no part of the curve represented by $y^2 = 5x^3$ can be situated on the left of the y-axis.

We notice again that, for any positive value of x, there will



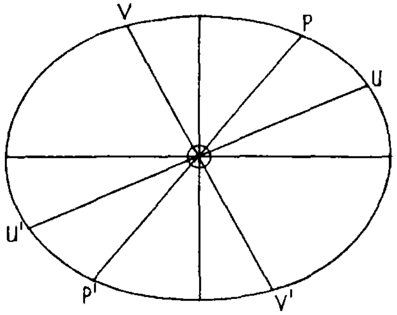
be two equal and opposite values of y. For example, if $x = 5$, $y = \pm 25$. And, generally, if two values of x and y such as x_1, y_1 satisfy the equation, then the values x_1 and $-y_1$ will also satisfy it. Thus the curve represented by this or any other equation in x and y, where $-y$ can be substituted for y without altering the form of the equation, must be symmetrical about the x-axis.

By similar reasoning, if we were dealing with the equation of a curve in which $-x$ could be substituted for x without altering the form of the equation, this would signify that the curve was symmetrical about the y-axis.

Again, in the equation $y^2 = 5x^3$, there is no limit to the positive value which x may have, and consequently none to the corresponding value of y. This signifies that the curve travels off to infinity in two directions, as shown in the diagram.

We will now consider the graph of such a curve as that represented by the equation

$$\frac{x^2}{16}+\frac{y^2}{9}=1.$$



The form of the equation is unaltered (1) when we put $-y$ for y, keeping x unchanged, (2) when we put $-x$ for x, keeping y unchanged.

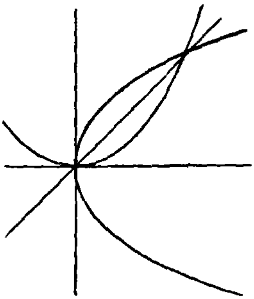
We infer, therefore, that the curve is symmetrical with regard to both of the axes.

Suppose that (x, y) was a point on the curve whose coordinates satisfied its equation, then clearly, if we put simultaneously $-x$ for x and $-y$ for y, the values $-x$ and $-y$ will satisfy it.

We infer from this that if any point $P\ (x, y)$ lies on the curve, there will be another point P' on it such that if POP' be drawn through the origin $OP' = OP$. This means that the curve is symmetrical about its centre.

Observe, however, that the fact of a curve being symmetrical about its centre does not imply that it must be symmetrical about each of the axes of coordinates. For instance, if the two perpendicular lines UOU', VOV' in the diagram had been taken as axes of coordinates, the curve would not be symmetrical about either of them, though it would still be symmetrical about the origin O.

Lastly, consider the curves represented by the equations $y^2 = 4x$, $x^2 = 4y$.



From what has been already discussed we see that both curves pass through the origin since $x = 0$, $y = 0$ satisfies both their equations.

That the curve represented by $y^2 = 4x$ is symmetrical with regard to the x-axis, has no portion of it in the second or third quadrants where x is negative, and goes off to infinity in two directions, one in the first and the other in the fourth quadrant.

Also that the curve represented by $x^2 = 4y$ is symmetrical with regard to the y-axis, has no portion of it in the third or fourth quadrants where y is negative, and goes off to infinity in two directions, one in the first and the other in the second quadrant.

But there is another fact to be observed, viz. that, in this case, either equation can be obtained from the other simply by interchanging x and y in its equation. This means that the situation of the curve represented by $y^2 = 4x$, relative to the x-axis, will be the situation of the curve represented by $x^2 = 4y$ relative to the y-axis. This is equivalent to saying that if the curve $y^2 = 4x$ were rotated through an angle of 90°, it would then occupy the position of the curve $x^2 = 4y$, or vice versa. Or again, it implies that the figure formed by both curves lies symmetrically about a line bisecting the angles between the axes in the first and third quadrants. The equation of such a line is $y = x$ because clearly the abscissa of any point on it is equal to the ordinate of that point.

We may therefore infer that, in general, if the equations of two curves are so related that if x and y be interchanged in the one we get the equation of the other, the situation of one curve with regard to one of the axes will be the same as the situation of the other with regard to the other axis, and that the figure formed by both curves will lie symmetrically about the line $y = x$ which bisects the angle between the axes.

By application of the foregoing principles we shall see, as we progress with this subject, that certain important geometrical facts about a curve whose equation is given can frequently be at once deduced from a simple inspection of that equation.

Practically all the geometrical properties of the various conic sections can be proved by analytical methods depending upon the use of equations. Theoretically, every geometrical theorem relating to them is capable of an analytical proof. It will be found, however, that in some cases a purely geometrical method leads to the easiest solution, while in others analysis is preferable. Often a combination of the two methods is the most advantageous. The student should therefore always have in his mind the geometrical significance of what he is doing. Wherever possible, the drawing of a rough diagram is a decided help in solving problems, unless their treatment is of a purely algebraical character.

TRANSFERENCE OF AXES

Naturally it is an advantage in all analytical work to have the equations of curves in their simplest form. Now as the form of the equation of any curve will largely depend upon the origin and axes with regard to which the x and y coordinates of any point on it are referred, a proper choice of origin and axes of coordinates is of prime importance.

Two very usual ways of making an equation simpler are (1) by transferring to a new origin but keeping the new axes through this in their original directions, (2) by keeping the origin fixed but turning the axes through a suitable angle.

Here it may be observed that throughout this book, except in the chapter where the use of oblique axes is especially considered, the axes of coordinates will always be assumed to be rectangular.

TRANSFERENCE TO PARALLEL AXES THROUGH A NEW ORIGIN

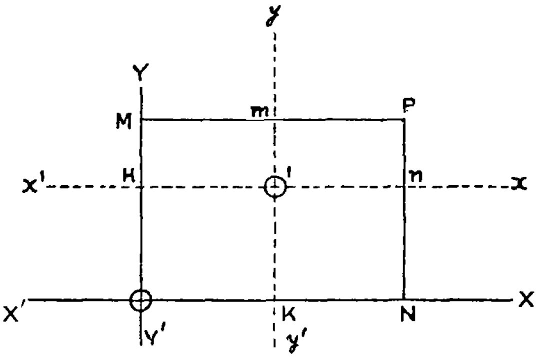
Let the original origin be O and the new origin O'.

Let the coordinates of O' be (h, k) referred to the original axes XOX', YOY' through O.

Let the coordinates of a point P be (x, y) with regard to the original axes, and (X, Y) with regard to parallel axes $xO'x'$, $yO'y'$ through O'.

Draw PmM, PnN perpendicular to OY and OX respectively, and cutting $O'y$ in m and $O'x$ in n.

Let $xO'x'$ cut OY in H and $yO'y'$ cut OX in K.



Then

$$x = ON = PM = Pm + mM = O'n + OK = X + h$$

and $y = OM = PN = Pn + OH = Y + k$.

∴ if in any equation we substitute $x+h$ for x and $y+k$ for y, we obtain another equation which represents the same curve as the former, but now with regard to parallel axes through the point (h, k).

Sometimes the point to be taken for the new origin is immediately obvious. Consider, for example, a curve whose equation is

$$a(x-3)^2 + b(y+1)^2 = c.$$

Here, by taking the point $(3, -1)$ as the new origin and referring to parallel axes through it, we get immediately by putting $x+3$ for x and $y-1$ for y

$$ax^2+by^2 = c,$$

which is a simpler equation to deal with.

Take now the curve whose equation is

$$4x^2-5xy-6y^2+5x+y+7 = 0.$$

Here we cannot see by mere inspection what is the best point to take as origin. We proceed, therefore, as follows:

Substitute $x+h$ for x and $y+k$ for y.

Then

$$4(x+h)^2-5(x+h)(y+k)-6(y+k)^2+5(x+h)+y+k+7 = 0.$$

As we aim probably at getting an equation which shall have no x term or y term in it, we choose h and k so that the coefficients of x and y in the transformed equation shall both be zero.

The coefficient of x in the transformed equation is $8h-5k+5$ and that of y is $-5h-12k+1$.

Hence $8h-5k+5 = 0$ and $-5h-12k+1 = 0$.

And solving these equations we have $h = -\frac{5}{11}$ and $k = \frac{3}{11}$. Therefore we take the point $(-\frac{5}{11}, \frac{3}{11})$ as our new origin and axes through it parallel to the original axes.

The transformed equation then becomes

$$4x^2-5xy-6y^2+4h^2-5hk-6k^2+5h+k+7 = 0,$$

which on the substitution of $h = -\frac{5}{11}$, $k = \frac{3}{11}$ gives

$$4x^2-5xy-6y^2+6 = 0,$$

a simpler form of equation than that with which we started.

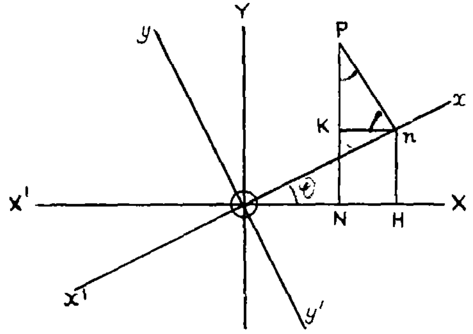
TRANSFERENCE TO AXES ROTATED THROUGH AN ANGLE θ KEEPING THE ORIGIN UNCHANGED

Let XOX', YOY' be the original axes and, keeping the origin fixed, rotate them through an angle θ into the positions xOx', yOy', where $\angle XOx = \theta$.

Let P be any point whose coordinates are (x, y) referred to the original axes and (X, Y) referred to the new ones.

Draw PN and Pn perpendicular to OX and Ox respectively, and also nH, nK perpendicular to OX and PN respectively. Then from the diagram

$$\angle KPn = 90^\circ - \angle KnP = \angle KnO = \angle XOx = \theta.$$



Hence

$$x = ON = OH - NH = On\cos\theta - Pn\sin\theta$$
$$= X\cos\theta - Y\sin\theta.$$

Also

$$y = PN = KN + PK = nH + PK = On\sin\theta + Pn\cos\theta$$
$$= X\sin\theta + Y\cos\theta.$$

$\therefore$ if in any equation we substitute $x\cos\theta - y\sin\theta$ for x and $x\sin\theta + y\cos\theta$ for y, we obtain another equation which represents the same curve as the former, but now with axes which have been rotated through an angle θ, the origin being the same as before.

As an illustration of this transformation, suppose we desired by a suitable rotation of axes to transform the equation $43x^2 + 14\sqrt{3}xy + 57y^2 = 4$ into an equation with no xy term in it.

Substituting $x\cos\theta - y\sin\theta$ for x and $x\sin\theta + y\cos\theta$ for y, the equation becomes

$$43(x\cos\theta - y\sin\theta)^2 + 14\sqrt{3}(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) +$$
$$+ 57(x\sin\theta + y\cos\theta)^2 = 4,$$

of which the coefficient of xy is

$$-86\sin\theta\cos\theta+14\sqrt{3}(\cos^2\theta-\sin^2\theta)+114\sin\theta\cos\theta.$$

Dividing throughout by $\cos^2\theta$ and equating this expression to zero, we have

$$-28\tan\theta = 14\sqrt{3}(1-\tan^2\theta);$$

whence $$\sqrt{3}\tan^2\theta-2\tan\theta-\sqrt{3} = 0,$$

or $$(\sqrt{3}\tan\theta+1)(\tan\theta-\sqrt{3}) = 0,$$

and $$\theta = 60^\circ \quad\text{or}\quad -30^\circ.$$

The axes could be turned through either of these angles.

Taking θ as -30°, the equation becomes

$$43(x\sqrt{3}+y)^2+14\sqrt{3}(x\sqrt{3}+y)(y\sqrt{3}-x)+57(y\sqrt{3}-x)^2 = 16;$$

whence $$144x^2+256y^2 = 16$$

and $$9x^2+16y^2 = 1.$$

TRANSFORMATION OF POLAR TO CARTESIAN COORDINATES

This *transformation* depends entirely upon the use of the relations $x = r\cos\theta$, $y = r\sin\theta$. The following examples will illustrate the methods employed.

(1) Transform the equation $r^2\cos 2\theta = a^2$ into a cartesian equation having the pole as its origin and the initial line as its x-axis.

$$r^2\cos 2\theta = r^2(\cos^2\theta-\sin^2\theta) = a^2.$$

$\therefore$ the required equation is $x^2-y^2 = a^2$.

(2) Similarly, transform the polar equation

$$r(\cos 3\theta+\sin 3\theta) = a\sin 2\theta.$$

$$\begin{aligned}\cos 3\theta+\sin 3\theta &= 4\cos^3\theta-3\cos\theta+3\sin\theta-4\sin^3\theta\\ &= (\cos\theta-\sin\theta)(4\cos^2\theta+4\sin^2\theta+4\sin\theta\cos\theta-3)\\ &= (\cos\theta-\sin\theta)(\cos^2\theta+\sin^2\theta+4\sin\theta\cos\theta).\end{aligned}$$

$$\therefore\ (r\cos\theta-r\sin\theta)(r^2\cos^2\theta+r^2\sin^2\theta+4r^2\sin\theta\cos\theta)$$
$$= r^3(\cos 3\theta+\sin 3\theta) = ar^2\sin 2\theta = 2ar^2\sin\theta\cos\theta.$$

$$\therefore\ (x-y)(x^2+y^2+4xy) = 2axy$$

is the equivalent cartesian equation.

Conversely,

(1) $y = mx$ becomes in polar coordinates

$$m = \frac{y}{x} = \tan\theta \quad \text{or} \quad \theta = \tan^{-1}m.$$

(2) $xy = c^2$ becomes

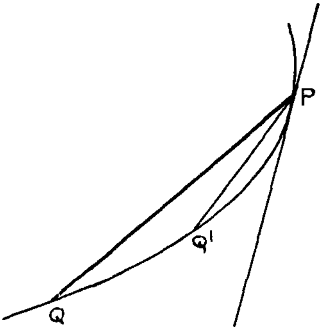
$$r^2\cos\theta\sin\theta = c^2 \quad \text{or} \quad r^2\sin 2\theta = 2c^2.$$

(3) $y^2 = 4ax$ becomes

$$r\sin^2\theta = 4a\cos\theta \quad \text{or} \quad r = 4a\cot\theta\operatorname{cosec}\theta.$$

THE TANGENT

Suppose that we take two points P and Q on a curve. Keeping P fixed, let Q move along the curve towards it. The chord PQ continually diminishes. When Q comes infinitely close to P and is ultimately about to coalesce with P, Q and P are then called consecutive points on the curve, and the line joining them, i e. the ultimate position of PQ, is the *tangent* to the curve at P.



A tangent to a curve may therefore be considered as the line joining two consecutive points on it.

A line through any point of a curve, perpendicular to the tangent at that point, is a *normal* to the curve at the point.

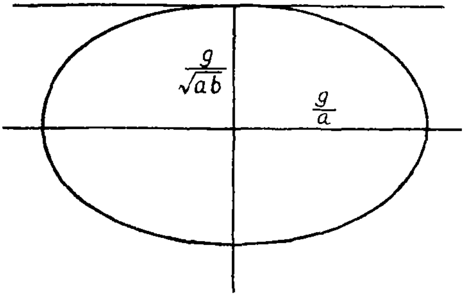
If two tangents are drawn from a point to a curve, the line joining their two points of contact with the curve is called the *polar* of the point with regard to the curve, and the point itself is the *pole* of such a line, with regard to the curve.

We will conclude this chapter with two instructive examples.

(1) Deduce some geometrical facts about the curve whose equation is $ax^2+by^2+2gx = 0$.

Firstly, $x = 0$, $y = 0$ satisfies the equation, so that the curve passes through the origin.

Next the substitution of $-y$ for y leaves the form of the equation unchanged, so that the curve is symmetrical with regard to the x-axis.



Rewrite the equation $\left(x+\frac{g}{a}\right)^2+\frac{b}{a}y^2 = \frac{g^2}{a^2}$.

Transferring to parallel axes through the point $\left(-\frac{g}{a}, 0\right)$, this becomes

$$a^2x^2+aby^2 = g^2, \tag{1}$$

the form of which would be unaltered if we put $-x$ for x keeping y unchanged, if we put $-y$ for y keeping x unchanged, or if we put simultaneously $-x$ for x and $-y$ for y.

Hence the curve is symmetrical about either axis or about the new origin, the point $\left(-\frac{g}{a}, 0\right)$ referred to the original axes.

Lastly, if we put $x = 0$ in equation (1), we have $y = \pm\sqrt{\frac{g}{ab}}$, and, if we put $y = 0$, $x = \pm\frac{g}{a}$.

The curve is therefore of the description shown in the diagram.

(2) If the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

represents two straight lines, find the coordinates of their point of intersection.

Suppose that the lines intersect at the point (x', y').

Transfer to parallel axes through (x', y') and we have

$$a(x+x')^2+2h(x+x')(y+y')+b(y+y')^2+$$
$$+2g(x+x')+2f(y+y')+c = 0.$$

Now we know that the equation of two straight lines through the origin must be of the form

$$Ax^2+2Hxy+By^2 = 0,$$

i.e. it has no x or y terms and no absolute term. So, in the transformed equation above, equate the coefficients of x and y to zero; then

$$ax'+hy'+g = 0 \quad \text{and} \quad hx'+by'+f = 0;$$

whence $$\frac{x'}{hf-bg} = \frac{y'}{gh-af} = \frac{1}{ab-h^2}.$$

The absolute term in the transformed equation, viz.

$$ax'^2+2hx'y'+by'^2+2gx'+2fy'+c, \qquad (1)$$

must be zero, because (x', y') must clearly satisfy the equation of the two straight lines.

The student familiar with the calculus will note that the equations $ax'+hy'+g = 0$ and $hx'+by'+f = 0$ are at once obtainable by partially differentiating the expression (1) successively with regard to x' and y'.

EXAMPLES 7

1. What is to be inferred about the curves whose equations are

(1) $x^2 = 4y+16$, (2) $x^2-9y^2 = 1$, (3) $xy = 4$,

(4) $(x-2)(y-3) = 6$, (5) $3x^2-7xy+2y^2 = 1$?

2. Show that the point $(4, 4)$ lies on the curve $y^2 = 4x$, that the point $(4, 5)$ lies outside it, and that the point $(4, 3)$ lies within it.

3. What can be deduced about the curve whose equation is $(y-2)^2 = 3x$, from inspection?

4. Show that if (x', y') is the point of intersection of two straight lines whose equation is $ax^2+2hxy+by^2+2gx+2fy+c = 0$, then $gx'+fy'+c = 0$.

5. Transfer the following equations to parallel axes:

(1) $9x^2+4y^2-6x-12y-9 = 0$, new origin $(\frac{1}{3}, \frac{3}{2})$.

(2) $24xy-42x-20y-29 = 0$, new origin $(\frac{5}{6}, \frac{7}{4})$.

(3) $5x^2+4xy-3y^2+14x-4y+3 = 0$, new origin $(-1, -1)$.

(4) $x^2+y^2+6x-4y-23 = 0$, new origin $(-3, 2)$.

6. Show that the curve represented by the equation

$$ax^2+2hxy+ay^2 = c$$

is symmetrical with regard to the lines $y = \pm x$ by turning the axes through an angle of 45°. How might this symmetry have been deduced at once?

7. If by turning the axes through an angle θ the equation $x^2+2\sqrt{3}xy+2y^2 = 6$ becomes an equation of the second degree in x and y of the form $ax^2+by^2 = c$, what must θ be?

8. What is the point of intersection of each pair of straight lines represented by the following equations?

(1) $6x^2-23xy-4y^2+29x-16y+20 = 0$.

(2) $9x^2+39xy+22y^2+33x+112y+10 = 0$.

(3) $6x^2\cos^2\theta-5xy\sin\theta\cos\theta-4y^2\sin^2\theta-10x\cos\theta+17y\sin\theta-4 = 0$.

9. Transfer the equation $x^2-y^2 = a^2$ to axes making 45° with the original axes, the origin remaining the same.

10. A line whose equation is $y = mx+\dfrac{a}{m}$ touches the curve whose equation is $y^2 = 4ax$. Find the equation of a line which will always touch the curve whose equation is $y^2 = 4a(x+a)$.

11. Prove that the four lines given by

$$p[(x+\alpha)^2-(y+\beta)^2]+q(x+\alpha)(y+\beta) = 0,$$
$$p[(x-\alpha)^2-(y-\beta)^2]+q(x-\alpha)(y-\beta) = 0$$

form a rectangle. Find also the equations of its diagonals.

12. Show that if by turning axes through an angle θ, the origin remaining fixed, the equation

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

is transformed into one having no xy term, then

$$\tan 2\theta = \frac{2h}{a-b}.$$

Formulae of Chapter III for reference

1. SYMMETRY RESULTS

If the form of the equation of a curve is unaltered when

(a) $-x$ is substituted for x in it, and y left unchanged:
It is symmetrical with regard to the x-axis.

(b) $-y$ is substituted for y in it, and x left unchanged:
It is symmetrical with regard to the y-axis.

(c) $-x$ is substituted in it for x, and $-y$ for y:
It is symmetrical with regard to the origin, and the origin is the centre of the curve.

If two equations are so related that one can be derived from the other by an interchange of x and y in either, the curves represented by them, taken as one figure, are symmetrical about the line $y = x$. Also one curve will be the same as the other rotated through an angle of 90°.

2. TRANSFERENCE OF AXES

To transform an equation to parallel axes through a new origin (h, k). Substitute $\mathbf{x+h}$ **for** $\mathbf{x}$ and $\mathbf{y+k}$ **for** $\mathbf{y}$ in the original equation.

To transform an equation to rectangular axes making each an angle of θ with the original axes. Substitute $\mathbf{x\cos\theta - y\sin\theta}$ **for** $\mathbf{x}$ and $\mathbf{x\sin\theta + y\cos\theta}$ **for** $\mathbf{y}$ in the original equation.

3. POLAR AND CARTESIAN EQUATIONS

To transform from one to another use the relations

$$\mathbf{x = r\cos\theta,} \qquad \mathbf{y = r\sin\theta.}$$

IV

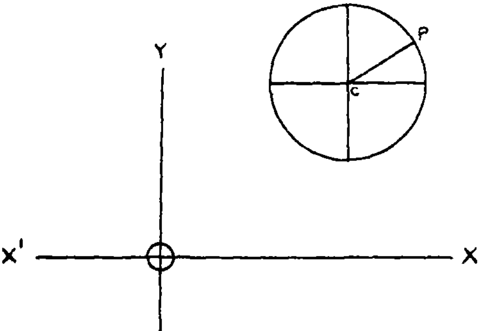
THE CIRCLE

IF a plane is drawn at right angles to the axis of a cone, its curve of intersection with the conical surface is a circle.

A *circle* is defined as the locus of a point which moves so that its distance from a fixed point called the centre is always the same.

1. The general equation of a circle

Let (x, y) be the coordinates of any point P on the circumference of a circle whose centre is C, a point (h, k) referred to a pair of rectangular axes XOX', YOY'.



Let the radius of the circle be a.

The square of the distance between P the point (x, y) and C the point (h, k) is $(x-h)^2+(y-k)^2$, which, by the definition of a circle, is always equal to a^2.

$\therefore$ the equation of the circle is

$$(x-h)^2+(y-k)^2 = a^2$$

or

$$x^2+y^2-2hx-2ky+h^2+k^2-a^2 = 0.$$

Write this equation as

$$x^2+y^2+2gx+2fy+c = 0. \qquad (1)$$

Then $-g = h$, $-f = k$, and $a^2 = h^2+k^2-c = g^2+f^2-c$.

We see then that in equation (1):

(*a*) the centre of the circle is the point $(\mathbf{-g, -f})$;

(*b*) the radius of the circle is $\sqrt{(\mathbf{g^2+f^2-c})}$.

Had the centre of the circle been taken as origin and two perpendicular diameters through it as axes of coordinates, then the square of the distance of any point (x, y) on the circumference from the origin being x^2+y^2, it follows that the equation of the circle would be

$$\mathbf{x^2+y^2 = a^2}.$$

By inspection of the equation $x^2+y^2 = a^2$ we observe:

(1) that if $-x$, $-y$ be substituted for x, y in it, the form of the equation is unaltered. This shows that a circle is symmetrical about its centre;

(2) that if $-x$ be substituted for x, keeping y unchanged, the form is also unaltered. This shows that the circle is symmetrical about the y-axis, i.e. about any diameter.

It should also be observed that, in the most general form of the equation of a circle, viz.

$$x^2+y^2+2gx+2fy+c = 0,$$

(1) *the coefficients of* x^2 *and* y^2 *are equal,* though they need not both be unity;

(2) *there is no* xy *term.*

Consider the equation $5x^2+5y^2-3x+4y+1 = 0$.

The coefficients of x^2 and y^2 are equal and there is no xy term. $\therefore$ the equation is that of a circle.

Rewrite it $x^2+y^2-\frac{3}{5}x+\frac{4}{5}y+\frac{1}{5} = 0$.

Then the coordinates of the centre are $\frac{3}{10}$, $-\frac{2}{5}$.

And the radius is of length $\sqrt{\{(\frac{3}{10})^2+(\frac{2}{5})^2-\frac{1}{5}\}} = \frac{1}{10}\sqrt{5}$.

2. Parametric coordinates

When the coordinates of any point on a curve can each be expressed in terms of a single variable, they are known as

Parametric Coordinates, and the variable is called a *Parameter.*

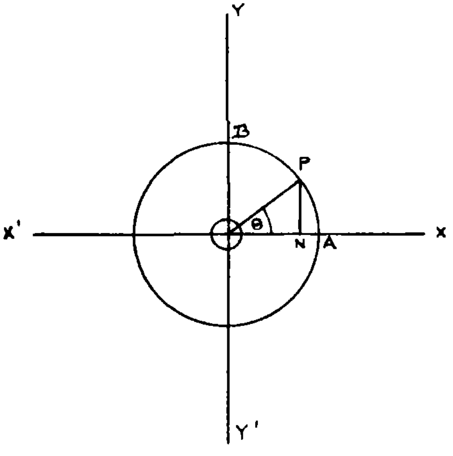
The coordinates of any point on the circle $x^2+y^2 = a^2$ may clearly be taken as $a\cos\theta$, $a\sin\theta$. For no matter what the value of the parameter θ may be, it is always true that

$$a^2\cos^2\theta+a^2\sin^2\theta = a^2,$$

i.e. the coordinates $a\cos\theta$, $a\sin\theta$ always satisfy the equation of the circle.

It is not essential that a parameter should always have a geometrical interpretation, but it is often quite easy to find one.

In the case of a circle, let P be any point on it referred to a pair of perpendicular diameters as axes of coordinates. Draw PN perpendicular to OA. Join OP. Let $\angle PON = \theta$.



Then the coordinates of P, viz. ON and PN, are respectively $OP\cos\theta$ and $OP\sin\theta$, i.e. $a\cos\theta$, $a\sin\theta$.

$\therefore$ $\angle PON$ is the geometrical representation of the parameter θ.

For brevity, the point on the circle $x^2+y^2 = a^2$ whose coordinates are $a\cos\theta$, $a\sin\theta$ will often be subsequently referred to simply as 'the point θ' on the circle.

3. Equation of a circle with a given diameter

Suppose A and B are two fixed points (h, k), (h', k').

Suppose that P is a point (X, Y) which moves so that PA is always perpendicular to PB.

The equation of PA, the line joining (X, Y) and (h, k), is

$$\frac{x-X}{X-h} = \frac{y-Y}{Y-k},$$

of which the gradient is $\dfrac{Y-k}{X-h}$.

Similarly, the gradient of PB is $\dfrac{Y-k'}{X-h'}$.

But, if PA and PB are at right angles, the product of the gradients of the two lines is -1.

$$\therefore \frac{(Y-k)(Y-k')}{(X-h)(X-h')} = -1,$$

i.e. $$(X-h)(X-h')+(Y-k)(Y-k') = 0.$$

$\therefore$ the locus of the point P is

$$(x-h)(x-h')+(y-k)(y-k') = 0,$$

which is clearly the equation of a circle.

We have accordingly proved by analysis that, if A and B are fixed points and the angle APB is right, P must be on a circle, and also have found a convenient form of the equation of that circle for writing down when the coordinates of A and B are given.

4. Equations of the chord and the tangent

Let P, Q be any two points (x', y'), (x'', y'') on the circle $x^2+y^2 = a^2$.

Then $$x'^2+y'^2 = a^2 = x''^2+y''^2.$$

$$\therefore (x'-x'')(x'+x'') = -(y'-y'')(y'+y''). \tag{1}$$

Again, the equation of the line PQ joining (x', y'), (x'', y'') is

$$\frac{x-x'}{x'-x''} = \frac{y-y'}{y'-y''}. \tag{2}$$

Combining equations (1) and (2) we have

$$(x-x')(x'+x'')+(y-y')(y'+y'') = 0,$$

i.e.

$$\begin{aligned} x(x'+x'')+y(y'+y'') &= x'^2+y'^2+x'x''+y'y'', \\ &= a^2+x'x''+y'y'', \end{aligned}$$

which is the equation of the chord PQ.

If $x' = x''$, $y' = y''$, P and Q become consecutive points and PQ is the tangent at the point (x', y').

So the equation of the tangent at the point (x', y') is

$$2xx'+2yy' = a^2+x'^2+y'^2 = a^2+a^2$$

or

$$\mathbf{xx'+yy' = a^2.}$$

Using parametric coordinates, suppose that P was the point θ and Q the point θ' on the circle $x^2+y^2 = a^2$. Then the equation of PQ is

$$\frac{x-a\cos\theta}{a(\cos\theta-\cos\theta')} = \frac{y-a\sin\theta}{a(\sin\theta-\sin\theta')},$$

and since

$$\cos\theta-\cos\theta' = -2\sin\frac{\theta+\theta'}{2}\sin\frac{\theta-\theta'}{2}$$

and

$$\sin\theta-\sin\theta' = 2\cos\frac{\theta+\theta'}{2}\sin\frac{\theta-\theta'}{2},$$

the equation becomes

$$\begin{aligned} x\cos\frac{\theta+\theta'}{2}+y\sin\frac{\theta+\theta'}{2} &= a\left(\cos\theta\cos\frac{\theta+\theta'}{2}+\sin\theta\sin\frac{\theta+\theta'}{2}\right) \\ &= a\cos\frac{\theta-\theta'}{2}. \end{aligned}$$

Putting $\theta = \theta'$, we have for the equation of the tangent at the point θ

$$\mathbf{x\cos\theta+y\sin\theta = a.}$$

Either of the forms of the equation of the tangent to the circle $x^2+y^2 = a^2$, just obtained, could be derived from the other by the use of the relations $x' = a\cos\theta$, $y' = a\sin\theta$.

Using the fact that $xx'+yy' = a^2$ is the tangent at the point

(x',y') to the circle $x^2+y^2 = a^2$, transfer to the point (g,f) as origin and to parallel axes through it. Then

$$(x+g)(x'+g)+(y+f)(y'+f) = a^2$$

is a tangent to the circle

$$(x+g)^2+(y+f)^2 = a^2,$$

i.e. to the circle

$$x^2+y^2+2gx+2fy+c = 0,$$

where $$c = g^2+f^2-a^2.$$

Hence the tangent to the circle $x^2+y^2+2gx+2fy+c = 0$ at the point (x',y') is

$$\mathbf{xx'+yy'+g(x+x')+f(y+y')+c = 0.}$$

For the student acquainted with the calculus the following method of obtaining the equation of the tangent at a point (x',y') is also available.

The equation of any line through the point (x',y') is

$$y-y' = m(x-x').$$

If this be the tangent at the point (x',y') of any curve whose equation is given, then m is the value of $\frac{dy}{dx}$ when $x \to x'$ and $y \to y'$.

Differentiating the equation $x^2+y^2 = a^2$, we have

$$2x+2y\frac{dy}{dx} = 0,$$

whence $$\frac{dy}{dx} = -\frac{x}{y}.$$

$\therefore$ the equation of the tangent at (x',y') is

$$y-y' = -\frac{x'}{y'}(x-x'),$$

i.e. $$xx'+yy' = x'^2+y'^2 = a^2.$$

The radius of the circle $x^2+y^2 = a^2$ through the point (x',y') has $\frac{x}{x'} = \frac{y}{y'}$ for its equation.

The gradient of this line is $\frac{y'}{x'}$, and the gradient of the tangent at (x', y') is $-\frac{x'}{y'}$.

$\therefore$ the product of these gradients is -1.

Hence the radius and the tangent at the point (x', y') of the circle are perpendicular to each other.

5. The normal

It has just been shown that the normal at any point of a circle is the radius through that point. We will, however, find its equation simply by assuming the definition that it is a line through (x', y') perpendicular to the tangent at this point.

The equation of the tangent is $xx'+yy' = a^2$.

That of a line through (x', y') perpendicular to it

$$xy'-yx' = x'y'-y'x' = 0,$$

i.e.
$$\frac{x}{x'} = \frac{y}{y'}.$$

The equation of the normal at the point θ is, similarly,

$$x\sin\theta - y\cos\theta = a\cos\theta.\sin\theta - a\sin\theta.\cos\theta$$
$$= 0,$$

i.e.
$$y = x\tan\theta.$$

6. Two tangents to a circle

If a point lies outside a circle, it is obvious that two tangents can be drawn from it to the circle. To a person ignorant of mathematical ideas, it might seem ludicrous to speak of tangents to a circle from a point within it. The fact is that these are imaginary; but, in analysis, imaginary quantities can be taken into account and thereby a certain continuity in thought is attained as well as some very interesting results in certain cases.

Consider a point whose coordinates are (h, k). The square of its distance from the centre of the circle whose equation is

$x^2+y^2 = a^2$, i.e. from the origin, is h^2+k^2. Hence, if the point (h, k) lies outside the circle,

$$h^2+k^2 > a^2.$$

If it lies on the circle, $h^2+k^2 = a^2$, so (h, k) must satisfy the equation $x^2+y^2 = a^2$.

If it lies inside the circle, then $h^2+k^2 < a^2$.

The equation of the tangent to the circle $x^2+y^2 = a^2$ at the point θ is

$$x\cos\theta+y\sin\theta = a.$$

If this passes through a point (h, k), we have

$$h\cos\theta+k\sin\theta = a$$

Put $\tan\frac{\theta}{2} = t$, then

$$\frac{h(1-t^2)}{1+t^2}+\frac{2kt}{1+t^2} = a,$$

i.e. $$t^2(a+h)-2kt+a-h = 0.$$

This is a quadratic equation in t and indicates that for any given point (h, k) there will be two values of t, i.e. two tangents can be drawn from (h, k) to the circle, which will be real, coincident, or imaginary, according as the roots of the quadratic in t are real, equal, or imaginary, i.e. according as

$$k^2 \gtreqless (a+h)(a-h)$$

or $$h^2+k^2 \gtreqless a^2,$$

the geometrical interpretation of this last result being whether the point (h, k) is outside, on, or within the circle.

In any given direction two real tangents can always be drawn to a circle.

Suppose that the line $y = mx+c$ has its gradient m fixed and that it touches the circle $x^2+y^2 = a^2$.

Then $mx-y+c = 0$ must be identical with

$$x\cos\theta+y\sin\theta-a = 0.$$

$$\therefore \frac{\cos\theta}{m} = \frac{-\sin\theta}{1} = -\frac{a}{c}.$$

$$\therefore\ 1 = \cos^2\theta+\sin^2\theta = \frac{a^2m^2}{c^2}+\frac{a^2}{c^2}.$$

$$\therefore\ c^2 = a^2(1+m^2).$$

$\therefore$ the lines $y = mx+a\sqrt{(1+m^2)}$, $y = mx-a\sqrt{(1+m^2)}$ are the only two parallel tangents of gradient m which can be drawn to the circle.

7. Tangential equation of a circle

We shall define the *tangential equation* of any curve as the condition expressed in terms of l, m, and n that the line whose equation is $lx+my+n = 0$ should touch it.

The equation of the tangent at the point θ on the circle $x^2+y^2 = a^2$ is
$$x\cos\theta+y\sin\theta-a = 0.$$

Identifying $xl+ym+n = 0$ with this, we have
$$\frac{\cos\theta}{l} = \frac{\sin\theta}{m} = -\frac{a}{n}.$$

$$\therefore\ 1 = \cos^2\theta+\sin^2\theta = \frac{a^2(l^2+m^2)}{n^2}.$$

$$\therefore\ n^2 = a^2(l^2+m^2)$$

is the tangential equation of the circle $x^2+y^2 = a^2$.

8. The polar of a point

As defined in Chapter III, the polar of a point with regard to a circle is the line joining the points of contact of the tangents drawn from the point to the circle.

These points of contact may be imaginary, but, as we shall see, no matter whether the point lies within or without the circle, its polar will always be real.

The student will therefore recognize that though two points may be imaginary, the line joining them can be real.

The point of intersection of the tangents to a circle $x^2+y^2 = a^2$ at the points θ_1, θ_2 is clearly obtained by solving the equations

$$x\cos\theta_1+y\sin\theta_1 = a \quad\text{and}\quad x\cos\theta_2+y\sin\theta_2 = a;$$

whence

$$\frac{x}{\sin\theta_1-\sin\theta_2} = \frac{y}{\cos\theta_2-\cos\theta_1} = \frac{-a}{\cos\theta_1\sin\theta_2-\cos\theta_2\sin\theta_1},$$

from which

$$x = \frac{a(\sin\theta_1-\sin\theta_2)}{\sin(\theta_1-\theta_2)} = \frac{\mathbf{a\cos\frac{\theta_1+\theta_2}{2}}}{\mathbf{\cos\frac{\theta_1-\theta_2}{2}}}$$

and

$$y = \frac{\mathbf{a\sin\frac{\theta_1+\theta_2}{2}}}{\mathbf{\cos\frac{\theta_1-\theta_2}{2}}}.$$

Using this result we can now find the equation of the polar of the point (h, k) with regard to the circle $x^2+y^2 = a^2$.

Suppose the two tangents from (h, k) to the circle touch at the points θ_1, θ_2 which may be real or imaginary.

Then $$h = \frac{a\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}} \quad\text{and}\quad k = \frac{a\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}.$$

Now the equation of the chord of the circle joining the points θ_1, θ_2 is

$$x\cos\frac{\theta_1+\theta_2}{2}+y\sin\frac{\theta_1+\theta_2}{2} = a\cos\frac{\theta_1-\theta_2}{2},$$

i.e. $$\mathbf{xh+yk = a^2},$$

which is the equation of the polar of the point (h, k) with regard to the circle no matter where this point may be situated. Further, so long as h and k are real, the equation of the polar $xh+yk = a^2$ must represent a real straight line.

Geometrically, the polar of a point with regard to a circle may be constructed thus:

The perpendicular from the origin, i.e. the centre of the circle, on the polar of a fixed point (h, k) whose equation is $xh+yk = a^2$, is of length

$$\frac{a^2}{\sqrt{(h^2+k^2)}}.$$

And the length of the line joining the points $(0, 0)$ and (h, k) is $\sqrt{(h^2+k^2)}$.

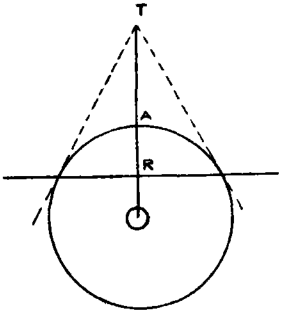
$\therefore$ the product of these lengths is a^2.

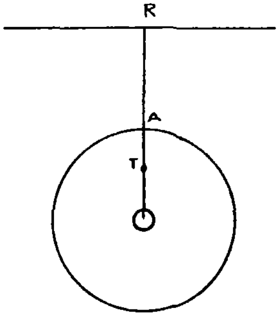
Hence, if T be a point whose polar is required to be constructed with regard to a circle of centre O and radius a, join OT.

Find a third proportional to OT and a, i.e. a length of $\dfrac{a^2}{OT}$.

Measure off on OT a distance $OR = \dfrac{a^2}{OT}$ producing OT if necessary to do so.

Through R draw a line perpendicular to OT; this line will clearly be the polar of the point T with regard to the circle.





If the polar of a point (h, k) with regard to a circle passes through another point (H, K), then the polar of (H, K) with regard to the circle will pass through the point (h, k).

The proof of this is as follows:

The polar of (h, k) with regard to the circle $x^2+y^2 = a^2$ is

$$xh+yk = a^2.$$

If this passes through (H, K), then

$$Hh+Kk = a^2.$$

The polar of (H, K) with regard to the circle is

$$xH+yK = a^2;$$

and since $$hH+kK = a^2,$$

$\therefore$ $x = h$, $y = k$ clearly satisfies the equation of the polar of (H, K) with regard to the circle, i.e. the point (h, k) lies on the polar of (H, K).

9. Properties of the circle

Some well-known geometrical properties of the circle will now be proved by the methods of analysis.

(1) The locus of the middle points of a system of parallel chords is the diameter perpendicular to them.

Take the centre of the circle as origin and a diameter parallel to the chords of the system as x-axis.

The equation of any one of these chords may be written as $y = K$.

Where it cuts the circle $x^2+y^2 = a^2$ we have

$$x^2 = a^2-K^2, \quad \text{i.e. } x = \pm\sqrt{(a^2-K^2)}.$$

This means that the algebraic sum of the abscissae of the two points in which $y = K$ cuts the circle is zero.

$\therefore$ the abscissa of the middle point of the line joining these two points is zero and consequently the middle point lies on the y-axis.

Hence the middle points of all chords of the parallel system lie on the y-axis.

In a circle, therefore, all chords parallel to a fixed diameter are bisected by the diameter perpendicular to it.

(2) The tangents at the extremities of a system of parallel chords meet on the diameter which bisects them.

As in the previous case, the tangents at the points

$$\{\sqrt{(a^2-K^2)}, K\} \quad \text{and} \quad \{-\sqrt{(a^2-K^2)}, K\}$$

are $$x\sqrt{(a^2-K^2)}+yK = a^2$$

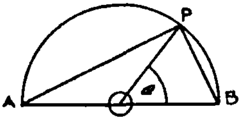
and $$-x\sqrt{(a^2-K^2)}+yK = a^2,$$

which clearly intersect at the point $x = 0$, $y = \frac{a^2}{K}$ which lies on the y-axis, i.e. the diameter bisecting all chords parallel to the fixed diameter taken as x-axis.

(3) To prove that the angle in a semicircle is a right angle.

Let P be any point θ on the semicircle $x^2+y^2 = a^2$ whose diameter is AB.

The parameters of P and A being θ and π respectively, the equation of PB is



$$x\cos\frac{\theta}{2}+y\sin\frac{\theta}{2} = a\cos\frac{\theta}{2}$$

and that of PA is

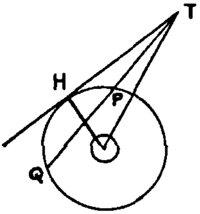
$$x\cos\frac{\pi+\theta}{2}+y\sin\frac{\pi+\theta}{2} = a\cos\frac{\pi-\theta}{2} = a\sin\frac{\theta}{2}.$$

The gradient of PB is $-\cot\frac{\theta}{2}$ and of PA is $-\cot\frac{\pi+\theta}{2}$.

But as $\cot\frac{\pi+\theta}{2} = -\tan\frac{\theta}{2}$, the product of these gradients is -1.

$\therefore$ PA and PB are perpendicular and $\angle APB = \frac{\pi}{2}$.

(4) To prove the 'rectangle' property, viz. that if from any point T outside a circle a line TPQ be drawn cutting the circle in P and Q, then $TP.TQ = TH^2$, where TH is the tangent from T to the circle.



Let the equation of the circle be

$$x^2+y^2 = a^2$$

and T the point (h, k).

The equation of any line through T can be written

$$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r.$$

And the coordinates of any point on it can be taken as $h+r\cos\theta$, $k+r\sin\theta$.

Substitute $x = h+r\cos\theta$, $y = k+r\sin\theta$ in the equation of the circle, then

$$(h+r\cos\theta)^2+(k+r\sin\theta)^2 = a^2,$$

i.e. $$r^2+2r(h\cos\theta+k\sin\theta)+h^2+k^2-a^2 = 0.$$

If r_1, r_2 are the roots of this equation,

$$\begin{aligned} r_1 r_2 &= h^2+k^2-a^2 \\ &= OT^2-OH^2 = TH^2. \end{aligned}$$

Hence, if TH be the tangent from the fixed point T to the circle,

$$TP.TQ = TH^2,$$

which proves the theorem.

10. Typical examples

Ex. 1. Find the coordinates of the centre and the length of the radius of the circle

$$2x^2+2y^2-10x-14y-13 = 0.$$

Rewrite the equation

$$x^2+y^2-5x-7y-\tfrac{13}{2} = 0.$$

By inspection the coordinates of the centre are $(\frac{5}{2},\frac{7}{2})$ and the radius $= \sqrt{\{(\frac{5}{2})^2+(\frac{7}{2})^2+\frac{13}{2}\}} = 5$.

Ex. 2. Find the equations of the tangents to the circle in the previous example which are parallel to the line $3x-4y = 5$.

The equation of the circle can be rewritten

$$(x-\tfrac{5}{2})^2+(y-\tfrac{7}{2})^2 = 25.$$

Transferring to parallel axes through the point $(\frac{5}{2},\frac{7}{2})$ the equation becomes

$$x^2+y^2 = 25.$$

The equations of the tangents to this circle of gradient $\frac{3}{4}$ are

$$y = \tfrac{3}{4}x\pm5\sqrt{(1+\tfrac{9}{16})},$$

i.e. $$y = \tfrac{3}{4}x\pm\tfrac{25}{4},$$

which, referred back to the original axes, are

$$y-\tfrac{7}{2} = \tfrac{3}{4}(x-\tfrac{5}{2})\pm\tfrac{25}{4},$$

i.e. $$8y-6x = 63 \quad\text{and}\quad 8y-6x+37 = 0.$$

Ex. 3. Find the equation of the circle circumscribing the triangle whose vertices are the points $(1,-2)$, $(3,4)$, $(-5,2)$.

Let the equation of the circle be $x^2+y^2+2gx+2fy+c = 0$.

Substituting the coordinates of the vertices in turn in this equation, we have

$$2g-4f+c = -5,$$
$$6g+8f+c = -25,$$
$$-10g+4f+c = -29;$$

whence, on solving, $g = \frac{8}{11}$, $f = -\frac{21}{11}$, $c = -\frac{155}{11}$. And the equation of the circle is

$$11x^2+11y^2+16x-42y-155 = 0.$$

Ex. 4. Find the equation of a circle which has the point $(3,-1)$ as its centre and touches the line $4x-3y = 9$.

Suppose the equation of the required circle is

$$(x-3)^2+(y+1)^2 = r^2.$$

Then r must equal the length of the perpendicular from $(3,-1)$ on $4x-3y-9 = 0$, which is $\frac{6}{5}$.

$\therefore$ the equation is

$$25(x-3)^2+25(y+1)^2 = 36,$$

i.e. $$25x^2+25y^2-150x+50y+214 = 0.$$

Ex. 5. Find the length of the chord in which the line $y = 3x+4$ cuts the circle $x^2+y^2 = 36$.

If Y be the middle point of a chord AB of a circle whose centre is C, then

$$AB^2 = 4AY^2 = 4(CA^2-CY^2).$$

The perpendicular from the centre of the circle on the line $y = 3x+4$ is of length $\frac{4}{\sqrt{10}}$.

$\therefore$ the length of the chord $= 2\sqrt{(36-\frac{16}{10})}$

$= \frac{4}{5}\sqrt{215}.$

Ex. 6. Find the pole of the line $6x-5y+5 = 0$ with regard to the circle $x^2+y^2-10x+12y-69 = 0$.

The polar of a point (x_1, y_1) with regard to this circle is

$$xx_1+yy_1-5(x+x_1)+6(y+y_1)-69 = 0,$$

i.e.
$$x(x_1-5)+y(y_1+6)-(5x_1-6y_1+69) = 0.$$

Identifying this equation with $6x-5y+5 = 0$, we have

$$\frac{x_1-5}{6} = \frac{y_1+6}{-5} = \frac{5x_1-6y_1+69}{-5};$$

whence $\qquad x_1 = -7, \qquad y_1 = 4.$

$\therefore$ the point $(-7, 4)$ is the pole of the line.

Ex. 7. The polar of the point (x', y') with respect to the circle $x^2+y^2 = 16$ touches the circle whose equation is $x^2+y^2-8x = 0$. Show that $y'^2+8x' = 16$.

The equation of the polar of (x', y') with regard to the circle $x^2+y^2 = 16$ is $xx'+yy' = 16$.

If this touches the circle $x^2+y^2-8x = 0$, then the equation

$$x^2+\left(\frac{16-xx'}{y'}\right)^2-8x = 0$$

must have equal roots.

$$\therefore\ 4\left(1+\frac{x'^2}{y'^2}\right)\frac{16^2}{y'^2} = \left(\frac{32x'}{y'^2}+8\right)^2 = 8^2\left(1+\frac{4x'}{y'^2}\right)^2,$$

i.e.
$$16(x'^2+y'^2) = y'^4+8y'^2x'+16x'^2$$

or
$$y'^2+8x' = 16.$$

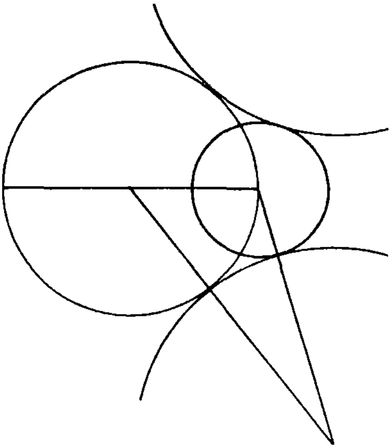
Ex. 8. Find the equations of the two circles which touch the circles $x^2+y^2 = a^2$ and $x^2+y^2-4ax = 0$ externally and whose radius is the arithmetic mean of their radii. [L.]

Equation $x^2+y^2-4ax = 0$ can be written

$$(x-2a)^2+y^2 = 4a^2,$$

of which the centre is the point $(2a, 0)$ and radius $2a$.

$\therefore$ the radius of the touching circle must be $\frac{3a}{2}$.



Let (x, y) be the coordinates of the centre of the required circle. Then, as in the diagram,

$$x^2+y^2 = \left(\frac{5a}{2}\right)^2, \qquad (x-2a)^2+y^2 = \left(\frac{7a}{2}\right)^2.$$

Solving, $$-2a(2x-2a) = 6a^2,$$

whence $$x = -\frac{a}{2}, \qquad y = \pm\sqrt{6}a.$$

$\therefore$ the equations of the two touching circles are

$$\left(x+\frac{a}{2}\right)^2+(y\pm\sqrt{6}a)^2 = \frac{9a^2}{4},$$

or $$x^2+y^2+ax\pm2\sqrt{6}ay+4a^2 = 0.$$

Ex. 9. Prove that the equation of any circle which passes through the points (h, k) and (h', k') can be put in the form

$$(x-h)\left(x-h'+\frac{\lambda}{h-h'}\right)+(y-k)\left(y-k'-\frac{\lambda}{k-k'}\right) = 0.$$

Show also that $\lambda = 0$ gives the smallest circle through the two given points. [B.]

Consider the equation

$$(x-h)\left(x-h'+\frac{\lambda}{h-h'}\right)+(y-k)\left(y-k'-\frac{\lambda}{k-k'}\right) = 0.$$

It is of the second degree, the coefficients of x^2 and y^2 are both unity, and there is no xy term.

$\therefore$ it is the equation of a circle.

$x = h$, $y = k$ obviously satisfy the equation.

Substitute $x = h'$, $y = k'$ and we get

$$(h'-h)\frac{\lambda}{h-h'}-(k'-k)\frac{\lambda}{k-k'} = -\lambda+\lambda = 0.$$

$\therefore$ the equation is that of a circle passing through the points (h, k), (h', k') no matter what λ may be.

The smallest circle going through the two points will clearly be that which has the line joining the points (h, k), (h', k') as its diameter.

Now the equation of a circle on the line joining (h, k), (h', k') as diameter is

$$(x-h)(x-h')+(y-k)(y-k') = 0,$$

which is what the given equation will reduce to if $\lambda = 0$.

$\therefore$ $\lambda = 0$ gives the smallest circle through the two given points.

Ex. 10. Prove that the equation of the circle through the points $(p, 0)$, $(q, 0)$, $(0, r)$ is

$$r(x^2+y^2)-r(p+q)x-(r^2+pq)y+pqr = 0.$$

If a circle passes through a fixed point, and the chord cut off from it by a fixed line is of constant length, find the locus of its centre. [O. and C.]

As in Ex. 3, let the equation of the circle through the three points be

$$x^2+y^2+2gx+2fy+c = 0.$$

Substituting in turn $x=p$, $y=0$; $x=q$, $y=0$; $x=0$, $y=r$, we have the equations

$$p^2+2gp+c = 0,$$
$$q^2+2gq+c = 0,$$
$$r^2+2fr+c = 0.$$

Solving which, we get $g = -\frac{p+q}{2}$, $c = pq$, $f = -\frac{r^2+pq}{2r}$.

Hence the equation of the circle is

$$r(x^2+y^2)-r(p+q)x-(r^2+pq)y+pqr = 0.$$

Suppose the circle in the second part of the question cuts the x-axis, taken as the fixed line, in the points $(p,0)$, $(q,0)$. Then p and q are variable, but by data $p-q = \lambda$, a constant.

Suppose the circle also goes through the fixed point $(0,r)$. Then its equation is as above. Let (X, Y) be its centre.

We have accordingly

$$2X = p+q, \qquad 2Y = r+\frac{pq}{r}.$$

Hence

$$\lambda^2 = (p-q)^2 = (p+q)^2-4pq = 4X^2-4r(2Y-r),$$

i.e. the locus of the centre is the curve whose equation is

$$4X^2-8rY+4r^2-\lambda^2 = 0.$$

Ex. 11. Find the equation of the two lines joining the origin to the points of intersection of the line $\frac{x}{a}+\frac{y}{b} = 1$ with the circle $x^2+y^2 = c^2$, and prove that if the line is a tangent to the circle, then $\frac{1}{a^2}+\frac{1}{b^2} = \frac{1}{c^2}$.

The equation of the lines joining the origin to the points of intersection of $\frac{x}{a}+\frac{y}{b} = 1$ and $x^2+y^2 = c^2$ is

$$x^2+y^2 = c^2\left(\frac{x}{a}+\frac{y}{b}\right)^2. \tag{1}$$

If the line is a tangent to the circle, then the two lines represented by equation (1) would be coincident. In that case

$$x^2\left(1-\frac{c^2}{a^2}\right)+y^2\left(1-\frac{c^2}{b^2}\right)-2c^2\frac{xy}{ab}$$

must be a perfect square.

$$\therefore\ \frac{c^4}{a^2b^2}=\left(1-\frac{c^2}{a^2}\right)\left(1-\frac{c^2}{b^2}\right),\quad \text{i.e. } \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}.$$

Ex. 12. Show that the locus of the points of intersection of perpendicular tangents to two concentric circles

$$x^2+y^2=a^2 \quad \text{and} \quad x^2+y^2=b^2$$

is

$$x^2+y^2=a^2+b^2.$$

Suppose the tangent at the point θ on the circle $x^2+y^2=a^2$ and the point ϕ on the circle $x^2+y^2=b^2$ are perpendicular, i.e.

$$x\cos\theta+y\sin\theta=a \quad \text{and} \quad x\cos\phi+y\sin\phi=b$$

are perpendicular.

$$\therefore\ \cot\theta\cot\phi=-1. \qquad \therefore\ \cos(\theta-\phi)=0.$$

$$\therefore\ \phi=\theta-90^\circ.$$

We therefore want the locus of the point of intersection of $x\cos\theta+y\sin\theta=a$ and $x\sin\theta-y\cos\theta=b$. Square and add these two equations. Then

$$x^2+y^2=a^2+b^2$$

is the required locus.

EXAMPLES 8

1. Find the equations of the circles in which
   (1) the radius is 5 and coordinates of the centre $(-3, 2)$,
   (2) the radius is $3\sqrt{2}$ and coordinates of the centre $(1, -4)$,
   (3) the radius is r and coordinates of the centre (a, b).

2. Write down the coordinates of the centres of the following circles and calculate their radii:
   (1) $x^2+y^2-8x+7=0$.
   (2) $x^2+y^2-3x+5y-1=0$.

(3) $x^2+y^2+ax+by+c = 0$.
(4) $x^2+y^2-2ax+2by+a^2 = 0$.

3. Find the equations of the circles which pass through the following points:

(i) $(0, 0)$, $(-5, 2)$, $(3, 1)$. (ii) $(3, 0)$, $(0, -2)$, $(-3, 4)$.
(iii) $(a-b, a+b)$, $(a+b, a-b)$, (a, b). (iv) $(a, 0)$, $(b, 0)$, $(0, c)$.

4. Show that the four points $(3, 4)$, $(3, -6)$, $(-1, 2)$, $(-1, -4)$ are concyclic and find the equation of the circle on which they lie.

5. Prove that the point $(5, -7\frac{1}{2})$ lies outside the circle whose equation is $x^2+y^2-4x+2y-44 = 0$.

6. The point $(1, -3)$ lies on the circle $x^2+y^2-6x+12y+32 = 0$. Find the coordinates of the other extremity of the diameter of the circle through this point.

7. Find the equation of the diameter of the circle

$$(x-2)^2+(y+4)^2 = 36$$

which passes through the point $(6, -6)$.

8. Find the equation of the circle which passes through the point $(-4, -2)$ and has the same centre as the circle whose equation is $x^2+y^2-4x-6y-23 = 0$.

9. Find the equation of the circle whose centre is at the point $(4, -3)$ and which passes through the origin.

10. Write down the equation of the circle of which the line joining the points $(-4, -3)$, $(-6, -2)$ is a diameter.

11. Prove analytically that a line can cut a circle in two points only, and that these may be imaginary.

12. What angle does the chord whose equation is

$$x\cos 37\tfrac{1}{2}^\circ + y\sin 37\tfrac{1}{2}^\circ = 16\cos 22\tfrac{1}{2}^\circ$$

subtend at the centre of the circle $x^2+y^2 = 256$?

13. Find the equations of the tangent and normal of the following circles at the point on them specified:

(1) $x^2+y^2 = 25$ at $(-4, 3)$.
(2) $x^2+y^2 = 16$ at $(2, 2\sqrt{3})$.
(3) $x^2+y^2 = 49$ at $(7\cos\alpha, 7\sin\alpha)$.
(4) $x^2+y^2-6x-4y-12 = 0$ at $(-1, 5)$.

(5) $x^2+y^2-8x-6y+20 = 0$ at $(2, 4)$.

(6) $2x^2+2y^2-5x+3y-35 = 0$ at $(-3, -2)$.

14. Show that the lines $x = 5$, $y = 7$ both touch the circle $x^2+y^2-4x-8y+11 = 0$.

15. If the line $y = mx$ touches the circle $x^2+y^2+2x-3y+2 = 0$, what values can m have?

16. Prove that the line $y = 3x+2\sqrt{10}$ touches the circle $x^2+y^2 = 4$.

17. Find the condition that the line $3x-4y+c = 0$ may touch the circle $x^2+y^2 = 9$.

18. Find the condition that the line $5x-y+c = 0$ may touch the circle $x^2+y^2-4x-6y-51 = 0$ by transferring both equations to parallel axes through the centre of the circle.

19. Show that the line $lx-my = 0$ touches the circle whose equation is $x^2+y^2-lx+my = 0$.

20. Find the equations of the circles which touch both axes of coordinates and pass through the point $(-4, -2)$.

21. Find the equations of the three circles which have their centres at the origin and touch respectively the lines

(1) $3x-4y = 10$, (2) $x\cos 60^\circ+y\sin 60^\circ = 6$, (3) $lx+my = 1$.

22. Find the equation of the circles which pass through the point $(-3, -5)$ and touch the line $4x+3y = 30$.

23. Find the equation of a circle which touches the y-axis at the point $(0, 5)$ and passes through the point $(3, 5)$.

24. Prove that the line $x(1-m^2)+2my = a(1+m^2)$ is a tangent to the circle $x^2+y^2 = a^2$.

25. Prove that the line $y = mx$ is a tangent to both the circles $x^2+y^2-2ax\sqrt{(1+m^2)}+a^2=0$, $x^2+y^2-2by\sqrt{(1+m^2)}+b^2m^2=0$; and deduce that the circles themselves will touch if $a = \pm mb$.

[O. and C.]

26. Prove that if the line $x\cos\alpha+y\sin\alpha = p$ touches the circle $(x-h)^2+(y-k)^2 = r^2$, then $h\cos\alpha+k\sin\alpha = p\pm r$. Explain the reason for the double sign.

27. Find the equation of the tangent to the circle

$$(x-h)^2+(y-k)^2 = a^2$$

at the point $(h+a\cos\alpha, k+a\sin\alpha)$.

28. Find the equations of the two tangents to the circle $x^2+y^2-6x-10y-30 = 0$ which make an angle of 60° with the x-axis.

29. Prove that the two circles which touch the x-axis and cut the y-axis at distances of a and b from the origin have as their equations $x^2+y^2\pm 2\sqrt{(ab)}x-(a+b)y+ab = 0$.

30. Show that the circle $x^2+y^2-2ax-2ay+a^2 = 0$ touches both axes of coordinates and find the coordinates of the points of contact.

31. Prove that the equations of the tangents to the circle whose equation is $x^2+y^2+2gx+2fy+c = 0$ parallel to the line joining the origin and its centre are

$$fx-gy = \pm\sqrt{\{(g^2+f^2)(g^2+f^2-c)\}}.$$

32. Find the coordinates of the points of intersection of the circle $x^2+y^2 = 25$ and the line $y = 2(x-1)$.

33. Prove that the product of the abscissae of the points in which the line $y = mx$ cuts the circle $x^2+y^2+2gx+2fy+c = 0$ is $\dfrac{c}{1+m^2}$.

34. Prove that if the line $y = mx+c$ cuts the circle $x^2+y^2 = a^2$, and if the circle intercepts a length of $2b$ on it, then

$$c^2 = (1+m^2)(a^2-b^2).$$

35. Write down the equation of the chord of a circle $x^2+y^2 = a^2$ joining two points whose radii make angles of 15° and 45° with the x-axis.

36. Find the equation of the chord of the circle $x^2+y^2 = 25$ which is bisected at the point $(3,-2)$.

37. PQ is a chord of the circle whose centre is at the origin O and radius 3. A is a fixed point on it. If $\angle POA = 120°$ and $\angle POQ = 45°$, what is the equation of PQ if OA is the x-axis?

38. If the tangents at (x_1, y_1), (x_2, y_2) on the circle

$$x^2+y^2+2gx+2fy+c = 0$$

are perpendicular, show that

$$x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+g^2+f^2 = 0.$$

39. Find the equation of the circle which passes through the points $(4,3)$, $(-2,-1)$ and has its centre on the line $3x-5y = 8$.

40. Find the equation of the two lines joining the origin to the points of intersection of the line $3x-y = 2$ and the circle

$$x^2+y^2-5x+8y+8 = 0.$$

41. Find the equation of the two lines joining the origin to the points of intersection of the line $\frac{x}{a}+\frac{y}{b} = 1$ with the circle

$$x^2+y^2 = c^2.$$

42. A circle cuts off lengths $2a$, $2b$ from the axes. Find the equation of the locus of its centre.

43. If a circle passes through a fixed point and cuts off a constant length on a fixed straight line, find the locus of its centre.

44. Find the angles at which the circle

$$(x-3)^2+(y-4)^2 = 36$$

cuts the x-axis.

45. The tangent to a given circle at any point P cuts the axes in L and M. Find the equation of the locus of the middle point of LM.

46. Find the equation of the tangent to the circle $x^2+y^2 = a^2$ which intercepts a distance l between the axes, in the positive quadrant.

47. Prove that the lines $y = \pm x \tan\alpha$ touch the circle

$$y^2\cos^2\alpha-x^2\sin^2\alpha+(x-a)^2 = 0$$

for all values of α.

48. Show that the locus of the point of intersection of perpendicular tangents to the circle $x^2+y^2 = a^2$ is the circle $x^2+y^2 = 2a^2$.

49. Find the locus of the point of intersection of two perpendicular straight lines of which one touches the circle $(x-a)^2+y^2 = b^2$ and the other touches the circle $(x+a)^2+y^2 = c^2$.

50. Show that the polar of a point (h,k) with regard to any circle such as $x^2+y^2-2gx = 0$, where g may vary, always passes through a fixed point.

51. By transferring to parallel axes through the point (g,f), find the equation of the polar of the point (h,k) with regard to the circle $x^2+y^2+2gx+2fy+c = 0$.

52. Write down the equation of the polar of the origin with regard to the circle $x^2+y^2+2gx+2fy+c = 0$.

53. Find the pole of the line $5x-12y = 20$ with regard to the circle $(x-2)^2+(y-1)^2 = 13$.

54. Find the equation of the polar of the point $(a+b, a-b)$ with regard to the circle $x^2+y^2-2ax-2by+c = 0$.

55. Prove that the area of the triangle formed by the two tangents to the circle $x^2+y^2 = a^2$ from the point (h, k), and their chord of contact, is

$$\frac{a(h^2+k^2-a^2)^{\frac{3}{2}}}{h^2+k^2}.$$

56. The ends of the base BC of a triangle ABC are the points $(a, 0)$, $(-a, 0)$; show that the locus of the orthocentre of the triangle is the circle whose equation is

$$x^2+y^2+2ay\cot A = a^2.$$

57. What is the locus of a point which moves so that its polar with regard to the circle $x^2+y^2 = a^2$ always passes through a fixed point (h, k)?

58. P is any point on the diameter AA' of a circle of radius a and BB' is the diameter perpendicular to AA'. If the perpendicular from A to BP cuts the polar of P with regard to the circle in Q, show that, referred to AOA', BOB' as axes, the locus of Q is $xy-ax+a^2 = 0$.

59. The polar of a point P with regard to the circle $x^2+y^2 = a^2$ is the line p, and the polar of Q is the line q. Show that the perpendiculars from P to q and from Q to p are in the same ratio as the distances of P and Q from the centre of the circle.

60. P and Q are two points on a diameter of a circle whose centre is O, such that $OP = OQ$. Find the locus of the points of intersection of the tangents to the circle from P and the polar of Q.

61. A point moves so that the rectangle of its distances from the two sides of a right-angled isosceles triangle exceeds the square of its distance from the hypotenuse by one-eighth the square of the hypotenuse. Find its locus.

62. Tangents are drawn from points on the circle $x^2+y^2 = 4a^2$ to the concentric circle $x^2+y^2 = a^2$. Prove that their chords of contact touch the circle $x^2+y^2 = \frac{1}{4}a^2$.

63. Find the equation of the circle which touches the circle $x^2+y^2 = 36$ at the point $(4, -2\sqrt{5})$ and is of radius 2.

64. Two circles touch each other, and one touches the x-axis at the point $(a, 0)$, the other touches the y-axis at the point $(0, b)$. Find the locus of the point where they touch each other.

65. Prove that the locus of the poles of the line $lx+my = 0$ with regard to circles of the system $y^2\cos^2\alpha-x^2\sin^2\alpha+(x-a)^2 = 0$ is $mx\sin^2\alpha+ly\cos^2\alpha = 0$. [O. and C.]

66. Show that a circle can be inscribed in the quadrilateral whose sides are

$$x = 0,\quad y = 0,\quad x\cos\alpha+y\sin\alpha = p,\quad x\cos\alpha'+y\sin\alpha' = p',$$

if $$p(1+\sin\alpha'+\cos\alpha') = p'(1+\sin\alpha+\cos\alpha).$$

67. A circle is drawn through the three points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, $(at_3^2, 2at_3)$. Show that it passes through the origin if

$$t_1+t_2+t_3 = 0.$$

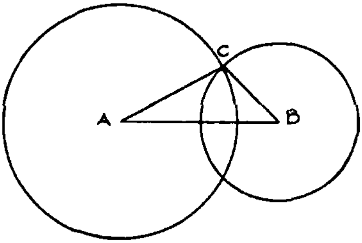
and in this case find the coordinates of its centre.

68. Find the equation of the circle which always touches the line whose equation is

$$(x-a)\cos\theta+(y-b)\sin\theta = r.$$

11. Orthogonal circles

Two circles are said to cut each other orthogonally, or to be *orthogonal*, when the tangents at their points of intersection are at right angles to each other.



Suppose that A and B are the centres of two orthogonal circles and that C is one of their points of intersection. As the tangents at C to the two circles are perpendicular, so must be their radii AC, BC, since each radius is perpendicular to its corresponding tangent. $\therefore \angle ACB$ is a right angle. Hence $AB^2 = AC^2+CB^2$.

To find the condition that two circles whose equations are

$$x^2+y^2+2gx+2fy+c = 0 \quad \text{and} \quad x^2+y^2+2Gx+2Fy+C = 0$$

should be orthogonal.

If A and B are the centres of the circles and C a point where they intersect, we have $AB^2 = AC^2+BC^2$.

Now A is the point $(-g, -f)$ and B the point $(-G, -F)$. Also $AC^2 = g^2+f^2-c$ and $BC^2 = G^2+F^2-C$.

$$\therefore \quad (-g+G)^2+(-f+F)^2 = g^2+f^2-c+G^2+F^2-C;$$

whence $$\mathbf{2Gg+2Ff-C-c = 0},$$

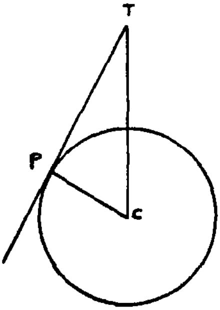
which is the requisite condition for the two circles to be orthogonal.

Note, as a particular case, that circles whose equations are of the form $x^2+y^2+2ax+c = 0$ and $x^2+y^2+2by-c = 0$ must always be orthogonal since the condition just proved is obviously satisfied by them.

12. The length of a tangent from a point to a circle

Let T be a point (h, k) and let the equation of the circle be

$$x^2+y^2+2gx+2fy+c = 0.$$



Draw a tangent TP to the circle. Let C be its centre and join CP, CT.

Then $TC^2 = CP^2+TP^2$.

The coordinates of C are $(-g, -f)$.

$$\therefore \quad (h+g)^2+(k+f)^2 = g^2+f^2-c+TP^2;$$

whence

$$TP^2 = h^2+k^2+2gh+2fk+c.$$

We get, therefore, this very simple rule for calculating the square of the length of a tangent from a point to a circle, viz. (1) Arrange the equation with all its terms on the left-hand side; (2) substitute in the terms of the expression on the left-hand side of the equation so arranged the coordinates of the point for x and y.

The square of the length of either tangent from a point to a circle is sometimes referred to as the 'power of the point' with regard to the circle.

13. The radical axis

The *Radical Axis* of two circles may be defined as the line joining their two points of intersection, real or imaginary.

Consider two circles whose equations are

$$x^2+y^2+2g_1x+2f_1y+c_1 = 0$$

and $$x^2+y^2+2g_2x+2f_2y+c_2 = 0.$$

The coordinates of a point common to both, i.e. of a point where they intersect, must satisfy both of these equations. $\therefore$ these coordinates must satisfy the equation obtained by subtracting, i.e.

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2 = 0.$$

This is the equation of a line going through the two points of intersection of the circles which, by definition, is their radical axis, for it is a linear equation and it is satisfied by the coordinates of each of the two points of intersection.

Let (X, Y) be any point on the radical axis of the two circles.

Then $$2X(g_1-g_2)+2Y(f_1-f_2)+c_1-c_2 = 0. \qquad (1)$$

The square of the tangent from (X, Y) to the first circle is

$$X^2+Y^2+2g_1X+2f_1Y+c_1,$$

and to the second circle

$$X^2+Y^2+2g_2X+2f_2Y+c_2.$$

The difference of these two expressions is

$$2(g_1-g_2)X+2(f_1-f_2)Y+c_1-c_2.$$

But equation (1) states that this is zero.

Hence *the lengths of the tangents to two circles from any point on their radical axis are equal.*

The equation of the radical axis of the two circles is

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2 = 0$$

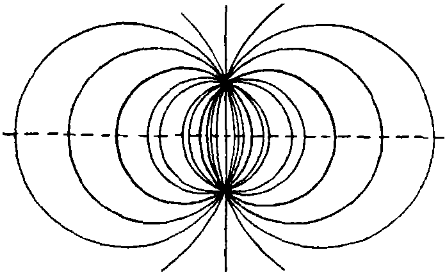
and that of the line joining their centres is

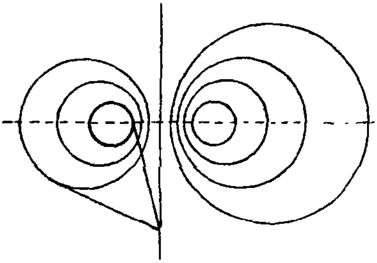
$$\frac{x+g_1}{-g_1+g_2} = \frac{y+f_1}{-f_1+f_2}.$$

Also, the product of the gradients of these two lines being -1, it follows therefore that *the radical axis of any two circles is always perpendicular to the line joining their centres.*

14. Coaxal circles

Coaxal circles are those of which any pair have the same radical axis.





Such a system of circles must therefore cut one another in the same two points, which may be real or may be imaginary. In the former case the system is called an intersecting one and, in the latter, a non-intersecting one.

The diagram above illustrates the two systems. In either, the line of centres of the circles must be perpendicular to the radical axis. That the centres all do lie on a line is clear from the fact that the line joining any two of them has been proved to be perpendicular to the radical axis. Thus, if A, B, and C are the centres of three coaxal circles, AB and AC are both perpendicular to the same line, the common radical axis. $\therefore$ A, B, C are collinear.

In dealing with coaxal circles the line of their centres is usually taken as the x-axis, and their common radical axis as the y-axis. The origin, where these two axes intersect, is sometimes called *the foot of the radical axis.*

The equation of any circle whose centre is on the x-axis can have no coefficient of the y term in it, because the ordinate of the centre is zero.

Hence the equation of any circle of a coaxal system can be written

$$\mathbf{x^2+y^2-2\kappa x+\delta^2 = 0},$$

where the axes of coordinates are the line of their centres and their common radical axis. In this equation κ is the abscissa of the centre and will be different for each circle of the system.

The tangent from the origin to any circle of this system is clearly of length $\sqrt{\delta^2}$, as is seen when zero is substituted for x and y in the expression on the left-hand side of the equation

$$x^2+y^2-2\kappa x+\delta^2 = 0.$$

But, we know that the length of the tangent from any point on the radical axis to any circle of the coaxal system must be the same.

Hence δ^2 will be a constant.

If δ^2 is positive, the length of the tangent from the foot of the radical axis to any circle of the coaxal system is real. Thus the foot of the radical axis must be outside all the circles and the system is a non-intersecting one.

On the other hand, if δ^2 is negative, the length of the tangent from the foot of the radical axis to any circle of the coaxal system is imaginary. In this case the foot of the radical axis must lie within each of the circles of the system, which is accordingly an intersecting one.

Consider δ^2 as positive. Then, from what has just been said, the equation $x^2+y^2-2\kappa x+\delta^2 = 0$ represents for different values of κ different circles of a non-intersecting coaxal system.

The radius of any circle of the system is $\sqrt{(\kappa^2-\delta^2)}$. So long as κ does not lie between δ and $-\delta$, this expression is real whether κ be positive or negative.

The circles become smaller as κ approaches δ. When $\kappa = \pm\delta$ we get two circles of zero radius. These are point circles of the system and are known as the *Limiting Points* of it.

They get this name because no circle of the coaxal system can have its centre between the limiting points $(\delta, 0)$ and $(-\delta, 0)$, and they are usually denoted as L and L'. They are, we observe, the same distance from the origin as the length of the tangent drawn from the origin to any circle of the system.

Interesting properties of the limiting points are the following:

(1) The polar of one limiting point $(\delta, 0)$ with regard to any circle of the coaxal system passes through the other limiting point $(-\delta, 0)$, and is perpendicular to the line of centres of the coaxal circles.

The equation of any circle of the system being

$$x^2+y^2-2\kappa x+\delta^2 = 0,$$

the equation of the polar of $(\delta, 0)$ with regard to this circle is

$$x\delta-\kappa(x+\delta)+\delta^2 = 0,$$

i.e.
$$(x+\delta)(\delta-\kappa) = 0.$$

But $\kappa \neq \delta$.
$$\therefore\ x+\delta = 0$$

is the equation of the polar of $(\delta, 0)$, and this is clearly a line through $(-\delta, 0)$ perpendicular to the x-axis, the line of centres of the coaxal circles.

(2) If a system of coaxal circles be drawn through the two limiting points of a non-intersecting coaxal system, any circle

of the former system will cut all the circles of the latter system orthogonally.

Let the equation of any circle of the non-intersecting coaxal system be
$$x^2+y^2-2\kappa x+\delta^2 = 0.$$

The equation of any circle passing through the limiting points and therefore having its centre on the y-axis will be
$$x^2+y^2-2\lambda y-\delta^2 = 0,$$
which is clearly satisfied by $x = \pm\delta$, $y = 0$, and also has the abscissa of its centre zero.

The squares of the radii of these two circles are respectively $\kappa^2-\delta^2$ and $\lambda^2+\delta^2$, of which the sum is $\kappa^2+\lambda^2$, i.e. the square of the distance between their centres.

$\therefore$ the two circles are orthogonal.

(3) If PQ be a common tangent to a pair of circles of a non-intersecting coaxal system, then it will subtend a right angle at either of the limiting points of the system.

Let two circles of the coaxal system have as their equations
$$x^2+y^2-2\kappa_1 x+\delta^2 = 0 \quad\text{and}\quad x^2+y^2-2\kappa_2 x+\delta^2 = 0.$$

Let a common tangent to the two circles cut the radical axis in the point $(0, \eta)$.

The length of the tangents from the point $(0, \eta)$ to either circle is $\sqrt{(\eta^2+\delta^2)}$. This is also the distance between the two points $(\delta, 0)$, $(0, \eta)$, or $(-\delta, 0)$, $(0, \eta)$.

$\therefore$ a circle with centre at the point $(0, \eta)$ and radius $\sqrt{(\eta^2+\delta^2)}$ goes through the limiting points and the points P, Q. As PQ is a diameter of this circle, it follows that PQ subtends a right angle at either of the limiting points.

15. The radical centre of three circles

We shall prove that the radical axes of any three circles, taken in pairs, are concurrent. The point where they meet is known as their *Radical Centre.*

For convenience we may take the centre of one of the circles as origin and a pair of perpendicular diameters through it as axes of coordinates.

The equations of the three circles may then be taken as

$$x^2+y^2 = a^2, \tag{1}$$

$$x^2+y^2+2g_1x+2f_1y+c_1 = 0, \tag{2}$$

$$x^2+y^2+2g_2x+2f_2y+c_2 = 0. \tag{3}$$

The radical axis of (1) and (2) found by subtraction is

$$2g_1x+2f_1y+c_1+a^2 = 0,$$

and that of (1) and (3) is

$$2g_2x+2f_2y+c_2+a^2 = 0.$$

Where these two lines intersect we have by subtraction

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2 = 0,$$

which is the equation of the radical axis of the circles (2) and (3).

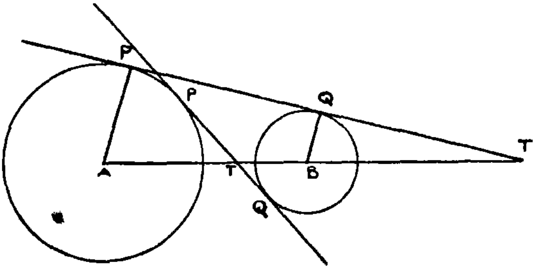
Hence the radical axis of circles (2) and (3) passes through the point of intersection of the radical axes of circles (1) and (2) and of circles (1) and (3).

Hence the three radical axes are concurrent.

16. Centres of similitude

Points on the line joining the centres of two circles which divide it internally and externally in the ratio of their radii are called the *Centres of Similitude* of the two circles.

If T and T' be the centres of similitude of two circles, the circle on TT' as diameter is called their *Circle of Similitude.*



Let a common tangent PQ to two circles whose centres are A and B meet the line AB or AB produced if necessary in T.

Then it is clear from the diagram that

$$TA : TB = AP : BQ = r_1 : r_2,$$

where r_1 and r_2 are the radii of the two circles.

$\therefore$ T is a centre of similitude.

The centres of similitude of two circles are therefore the two points on the line of centres of the circles where common tangents to them meet.

Let two circles, referred to their line of centres and radical axis as axes of coordinates, have as their equations

$$x^2+y^2-2\kappa_1 x+\delta^2 = 0 \quad \text{and} \quad x^2+y^2-2\kappa_2 x+\delta^2 = 0.$$

Then the centres of the circles being the points $(\kappa_1, 0)$, $(\kappa_2, 0)$, the coordinates of the centres of similitude are, by definition,

$$\left(\frac{r_2\kappa_1+r_1\kappa_2}{r_2+r_1}, 0\right) \quad \text{and} \quad \left(\frac{r_2\kappa_1-r_1\kappa_2}{r_2-r_1}, 0\right),$$

r_1 and r_2 being the radii of the circles.

The equation of the circle on the line joining the two centres of similitude as diameter is (Art. 3)

$$\left(x-\frac{r_2\kappa_1+r_1\kappa_2}{r_2+r_1}\right)\left(x-\frac{r_2\kappa_1-r_1\kappa_2}{r_2-r_1}\right)+y^2 = 0,$$

i.e.
$$x^2+y^2-2\frac{r_2^2\kappa_1-r_1^2\kappa_2}{r_2^2-r_1^2}x+\frac{r_2^2\kappa_1^2-r_1^2\kappa_2^2}{r_2^2-r_1^2} = 0.$$

But as $\delta^2 = \kappa_1^2-r_1^2 = \kappa_2^2-r_2^2$, we have

$$\kappa_1^2 r_2^2-\kappa_2^2 r_1^2 = (r_1^2+\delta^2)r_2^2-(r_2^2+\delta^2)r_1^2 = \delta^2(r_2^2-r_1^2).$$

$\therefore$ the equation of the circle of similitude can be written

$$x^2+y^2-2\frac{r^2\kappa_1-r_1^2\kappa_2}{r_2^2-r_1^2}x+\delta^2 = 0,$$

the form of which shows that it is coaxal with the two given circles.

17. Equation of tangents to a circle

Let (h, k) be any point outside the circle $x^2+y^2 = a^2$.

Also let (X, Y) be any other point.

The coordinates of the point dividing the line joining (h, k), (X, Y) in the ratio $p : q$ are

$$\frac{pX+qh}{p+q}, \qquad \frac{pY+qk}{p+q}.$$

Substituting these coordinates in the equation of the circle, we get

$$(pX+qh)^2+(pY+qk)^2 = (p+q)^2a^2,$$

which may be written

$$\left(\frac{p}{q}\right)^2(X^2+Y^2-a^2)+2\left(\frac{p}{q}\right)(Xh+Yk-a^2)+h^2+k^2-a^2 = 0.$$

The two values of $\left(\frac{p}{q}\right)$ obtained from this equation give the two points at which the line joining the points (X, Y), (h, k) cuts the circle.

But if the two values of $\frac{p}{q}$ are equal, this line becomes a tangent to the circle. In this case

$$(Xh+Yk-a^2)^2 = (X^2+Y^2-a^2)(h^2+k^2-a^2)$$

becomes the locus of the point (X, Y) when it lies on a tangent from the point (h, k) to the circle. Hence the equation of the tangents from (h, k) to the circle $x^2+y^2 = a^2$ is

$$\mathbf{(xh+yk-a^2)^2 = (x^2+y^2-a^2)(h^2+k^2-a^2)}.$$

The form of this equation should be noticed.

(1) The left-hand side of it corresponds to the form of the equation of the polar of a point (h, k) with regard to the circle.

(2) The first factor on the right-hand side of the equation corresponds to the form of the equation of the circle itself.

(3) The second factor on the right-hand side of the equation corresponds to the form of the equation of the circle itself when h and k are substituted for x and y in it.

18. Equation of common tangents

The method of obtaining equations of common tangents of two circles is:

(1) Find the coordinates of the centres of similitude.

(2) Find the equations of the tangents from these points to either of the circles.

This method is illustrated by the following example.

Take the two circles whose equations are

$$x^2+y^2 = 1 \quad \text{and} \quad x^2+y^2-2x-6y+6 = 0.$$

The radii of the circles are 1 and 2.

The coordinates of their centres the points $(0,0)$, $(1,3)$.

The coordinates of their centres of similitude the points $(\frac{1}{3},1)$, $(-1,-3)$.

The equation of the tangents from the point $(\frac{1}{3},1)$ to the circle $x^2+y^2 = 1$ is

$$\left(\frac{x}{3}+y-1\right)^2 = (x^2+y^2-1)\tfrac{1}{9},$$

i e. $$4y^2+3xy-3x-9y+5 = 0,$$

or $$(4y+3x-5)(y-1) = 0.$$

Similarly, the equation of the tangents from the point $(-1,-3)$ is

$$(-x-3y-1)^2 = (x^2+y^2-1)9,$$

i.e. $$4x^2-3xy-x-3y-5 = 0,$$

or $$(4x-3y-5)(x+1) = 0.$$

Hence the equation of the four common tangents of the two circles are $x+1 = 0$, $y-1 = 0$, $3x+4y-5 = 0$, and $4x-3y-5 = 0$.

19. Polar equation of the circle

Let O be the origin and OX the initial line.

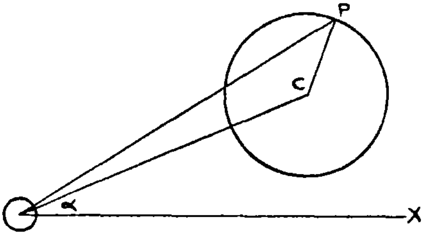
Let P be a point on a circle whose centre is C and radius a. Let $\angle COX = \alpha$ and $\angle POX = \theta$. Also let $OC = d$. Then

$$CP^2 = OC^2+OP^2-2OC.OP\cos\angle POC,$$

i.e. $$a^2 = d^2+r^2-2dr\cos(\theta-\alpha),$$

which is the polar equation of a circle, and may be rewritten

$$\mathbf{r^2-2r\,.\,d\cos(\theta-\alpha)+d^2-a^2 = 0.}$$



Special cases arise

(1) When OC is taken as the initial line and $\alpha = 0$. The equation then becomes

$$r^2-2rd\cos\theta+d^2-a^2 = 0.$$

(2) If O is on the circumference of the circle, $a = d$, and the equation becomes
$$\mathbf{r = 2a\cos\theta.}$$

When two points P and P' on a straight line through a fixed point O are so related that $OP' = \kappa OP$, it is clear that P' will trace out a curve similar to that traced out by P.

For instance, if $r' = \kappa r$ and we put $\frac{r'}{\kappa}$ for r in the equation of the circle $r^2-2dr\cos(\theta-\alpha)+d^2-a^2 = 0$, we get

$$r'^2-2\kappa dr'\cos(\theta-\alpha)+\kappa^2(d^2-a^2) = 0,$$

which is clearly the equation of a concentric circle whose centre is κd distant from the origin and whose radius is κa.

Again, if two points P and P' on a straight line through a fixed point O are so related that $OP.OP' = \kappa^2$, where κ is a constant, the curve traced out by P' is called the inverse of that traced out by P.

To find the inverse of a circle whose centre is C and radius a with regard to a fixed point O, where $OC = d$.

Take O as the pole and OC as the initial line.

The equation of the circle is then

$$r^2-2rd\cos\theta+d^2-a^2=0.$$

Putting in this $r=\dfrac{\kappa^2}{r'}$, where $r=OP$ and $r'=OP'$, we have

$$\kappa^4=r'^2(a^2-d^2)+2\kappa^2dr'\cos\theta,$$

which, transferred to cartesian coordinates and taking O as origin, the initial line OC as x-axis, and a line through O perpendicular to OC as y-axis, becomes

$$x^2+y^2-\frac{2\kappa^2d}{d^2-a^2}x+\frac{\kappa^4}{d^2-a^2}=0.$$

This is the equation of a circle whose centre lies on OC, the x-axis, and at a distance along it of $\dfrac{\kappa^2d}{d^2-a^2}$ from O. The radius of the inverse circle is also

$$\sqrt{\left(\frac{\kappa^4d^2}{(d^2-a^2)^2}-\frac{\kappa^4}{d^2-a^2}\right)}=\frac{\kappa^2a}{d^2-a^2}.$$

Had the point O been taken on the circumference of the given circle, so that $d=a$, the equation of the inverse would become $2ax=\kappa^2$.

That is to say, the inverse of a circle, with regard to a given point on its circumference, is the straight line perpendicular to the diameter through that point and passing through the point on that diameter which is the inverse of its other extremity with regard to the given point.

20. Typical examples

Ex. 1. Find the equation of the circle which cuts orthogonally the circle $x^2+y^2-6x+4y-3=0$, passes through $(3,0)$, and touches the axis of y. [O. and C.]

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$.

The orthogonal condition gives $-6g+4f+3-c=0$.

Since $x=0$ touches the circle, $y^2+2fy+c=0$ has equal roots. $\therefore f^2 = c$. Again, $(3, 0)$ satisfies the equation of the circle. $\therefore 9+6g+c = 0$; whence $f = g = -3$ and $c = 9$.

Hence the required equation is $x^2+y^2-6x-6y+9 = 0$.

Ex. 2. Prove that the lengths of the tangents from any point on the circle $15x^2+15y^2-48x+64y = 0$ to the two circles

$$5x^2+5y^2-24x+32y+75 = 0$$

and

$$5x^2+5y^2-48x+64y+300 = 0$$

are in the ratio of the radii of the circles. [C.]

The squares of the radii of the circles are $12^2+16^2-75 = 325$ and $24^2+32^2-300 = 1300$.

If (h, k) be a point on the first circle, the squares of the lengths of the tangents to the other two are in the ratio

$$\frac{5(h^2+k^2)-24h+32k+75}{5(h^2+k^2)-48h+64y+300} = \frac{48h-64k-72h+96k+225}{48h-64k-144h+192k+900}$$

since $15(h^2+k^2) = 48h-64k$.

And this ratio becomes

$$\frac{-24h+32k+225}{-96h+128k+900} = \tfrac{1}{4}.$$

Hence the ratio of the tangents and of the radii is $\tfrac{1}{2}$.

Ex. 3. Prove that the circles whose equations are given as

$$x^2+y^2+8 = 2\kappa(x+y-1)$$

for various values of κ form a coaxal system, and find the coordinates of the limiting points of the system. [J.]

Transform to axes $x+y-1 = 0$ and $x-y = 0$, i.e. put $X = x+y-1$ and $Y = x-y$.

Then $X+Y = 2x-1$, $X-Y = 2y-1$, and the equation becomes

$$(X+Y+1)^2+(X-Y+1)^2+32 = 8\kappa X,$$

i.e.

$$(X+1)^2+Y^2+16 = 4\kappa X$$

or

$$X^2+Y^2-2X(2\kappa-1)+17 = 0,$$

which is of the form

$$X^2+Y^2-2\lambda X+17 = 0,$$

and therefore the system is a coaxal one with $X = 0$, i.e. $x+y-1 = 0$, for its radical axis and $Y = 0$, i.e. $x-y = 0$, for its line of centres.

The coordinates of the limiting points will be $X = \pm\sqrt{17}$, $Y = 0$.

$$\therefore\ x = \frac{\pm\sqrt{17}+1}{2} = y,$$

i.e. the coordinates of the limiting points referred to the original axes and origin are

$$\frac{1\pm\sqrt{17}}{2},\ \frac{1\pm\sqrt{17}}{2}.$$

Ex. 4. Show that in the case of coaxal circles the lengths of the tangents from a point on one of them to any two others are in a constant ratio.

Let the three coaxal circles be $x^2+y^2-2\kappa_1 x+\delta^2 = 0$, $x^2+y^2-2\kappa_2 x+\delta^2 = 0$, and $x^2+y^2-2\kappa_3 x+\delta^2 = 0$, and let (α, β) be a point on the first.

Then $\alpha^2+\beta^2-2\kappa_1\alpha+\delta^2 = 0$.

The square of the length of the tangent from (α, β) to the second circle is $\alpha^2+\beta^2-2\kappa_2\alpha+\delta^2 = 2\kappa_1\alpha-2\kappa_2\alpha = 2\alpha(\kappa_1-\kappa_2)$.

Similarly, the square of the length of the tangent from (α, β) to the third circle is $2\alpha(\kappa_1-\kappa_3)$.

$\therefore$ the lengths of the tangents are in the ratio

$$\sqrt{(\kappa_1-\kappa_2)} : \sqrt{(\kappa_1-\kappa_3)},$$

which is independent of (α, β) and therefore a constant ratio.

Ex. 5. Show that if S stands for $x^2+y^2+2gx+2fy+c$ and S' stands for $x^2+y^2+2g'x+2f'y+c'$, then $S-\lambda S' = 0$ is the equation of a circle passing through the points common to $S = 0$ and $S' = 0$.

$S-\lambda S'$ is the equation of a circle because the coefficients of x^2 and y^2 are both equal to $1-\lambda$ and there is no xy term.

Further, if the coordinates of any point satisfy $S = 0$ and $S' = 0$, it is a point common to these circles, i.e. a point of their intersection.

$\therefore$ the same coordinates must satisfy $S-\lambda S' = 0$, i.e. the circle $S-\lambda S'=0$ passes through the points of intersection of $S=0$ and $S'=0$.

Ex. 6. If Q is a point on the polar of P with regard to a given circle, prove that the circle on PQ as diameter cuts the given circle orthogonally.

Let the equation of the circle be $x^2+y^2=a^2$.

Let P be a point (h,k) and let Q be the point (H,K) on the polar of P.

Then (H,K) satisfies the equation $xh+yk=a^2$.

$$\therefore\ Hh+Kk=a^2.$$

The equation of the circle on PQ as diameter is

$$(x-h)(x-H)+(y-k)(y-K)=0,$$

i.e. $$x^2+y^2-x(h+H)-y(k+K)+Hh+Kk=0,$$

which is clearly orthogonal to $x^2+y^2=a^2$, since this last equation has no coefficients of the x and y terms and

$$Hh+Kk-a^2=0.$$

Ex. 7. Find the equation of the inscribed circle of the triangle whose sides have the equations

$$4x+3y=1,\quad -4x+3y=1,\quad y=3. \qquad \text{[J.]}$$

Solving the equations, the three vertices are the points $(0,\frac{1}{3})$, $(2,3)$, $(-2,3)$, the triangle is therefore isosceles, and the bisector of the angle at the vertex $(0,\frac{1}{3})$ is the y-axis $x=0$. The bisector of the angle between the two lines $-y+3=0$, $4x-3y+1=0$ is the line

$$-y+3=-\frac{4x-3y+1}{5},$$

the minus sign being taken because the origin lies outside the angle.

Where $x=0$ and this line intersect we have $y=2$.

$\therefore$ the inscribed centre is the point $(0,2)$, and the radius of the circle is clearly $3-2=1$.

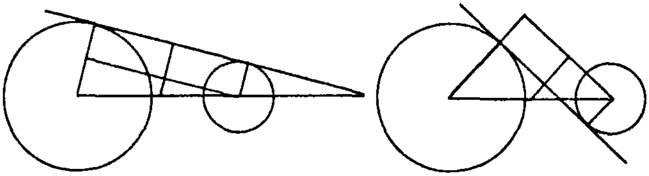
$\therefore$ its equation is $$x^2+(y-2)^2=1,$$

i.e. $$x^2+y^2-4y+3=0.$$

Ex. 8. Two circles have their centres at the points $(a, 0)$, $(-a, 0)$ and the equations of their two external common tangents are $x\cos\alpha \pm y\sin\alpha = a\cos\beta$; prove that the equations of their internal common tangents are $x\cos\beta \pm y\sin\beta = a\cos\alpha$. [O. and C.]

Let r_1 and r_2 be the radii of the circles.

The perpendicular from the origin on the external tangent



$x\cos\alpha+y\sin\alpha = a\cos\beta$ is of length $a\cos\beta$, it is also of length $\frac{r_1+r_2}{2}$. $\therefore\ r_1+r_2 = 2a\cos\beta$.

Let $x\cos\theta+y\sin\theta = \frac{r_2-r_1}{2}$ be the equation of an internal tangent. Then $2a\cos\theta = r_2+r_1 = 2a\cos\beta$. $\therefore\ \theta = \beta$.

And $2a\cos\alpha = r_2-r_1$.

$\therefore$ the equation of an internal tangent is

$$x\cos\beta+y\sin\beta = a\cos\alpha$$

and that of the other internal tangent must be

$$x\cos\beta-y\sin\beta = a\cos\alpha.$$

Ex. 9. Prove that the tangent at the point θ to the circle whose polar equation is $r = d\cos(\theta-\alpha)$, where $\alpha < \theta < \frac{\pi}{4}+\frac{\alpha}{2}$ cuts the initial line at a distance of $\frac{d\cos^2(\theta-\alpha)}{\cos(2\theta-\alpha)}$ from the origin. [B.]

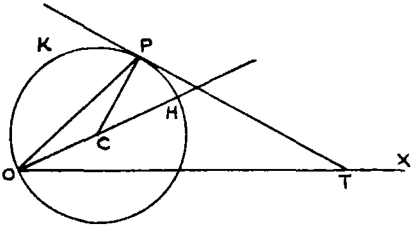
In the diagram $\angle POX = \theta$, OX being the initial line. The tangent at P cuts OX in T. C is the centre of the circle. OC, CP are joined.

$$\angle OPT = \angle CPT + \angle CPO = 90^\circ + \theta - \alpha \text{ since}$$

$$\angle CPO = \angle COP.$$

$$\therefore\ \angle OTP = 180^\circ - \angle OPT - \angle POT = 90^\circ - (2\theta - \alpha).$$

$$\therefore\ \frac{OT}{OP} = \frac{\sin\angle OPT}{\sin\angle OTP} = \frac{\sin(90^\circ + \theta - \alpha)}{\sin(90^\circ - 2\theta + \alpha)} = \frac{\cos(\theta - \alpha)}{\cos(2\theta - \alpha)}.$$



But $OP = d\cos(\theta - \alpha)$, where d is the diameter of the circle.

$$\therefore\ OT = \frac{d\cos^2(\theta - \alpha)}{\cos(2\theta - \alpha)}.$$

For the tangent at P to be between H and K in the figure it is clear that $\alpha < \theta < \frac{1}{2}\left(\frac{\pi}{2} + \alpha\right)$.

Ex. 10. Circles are drawn with their centres on the positive part of the x-axis, so as to touch the circle $x^2 + y^2 = a^2$ externally. Prove that the locus of the point of contact of such a circle with a common tangent to it and the circle $x^2 + y^2 = a^2$ is the cubic curve whose equation is

$$y^2(3a - x) = (x - a)^3. \qquad \text{[O.]}$$

Let the common tangent touch one circle of the system in P and the circle $x^2 + y^2 = a^2$ in Q. Let C be the centre of the variable circle. Join CP and OQ. From C draw CR parallel to the common tangent to meet OQ in R.

Let P be the point (X, Y), and suppose that

$$x\cos\theta + y\sin\theta = a$$

is a tangent also to the variable circle, of radius r suppose.

Then $X = a+r+r\cos\theta, \qquad Y = r\sin\theta.$

Also $(a+r)\cos\theta = OR = a-r.$

$$\therefore \frac{(X-a)^2}{Y^2} = \left(\frac{1+\cos\theta}{\sin\theta}\right)^2 = \cot^2\frac{\theta}{2},$$

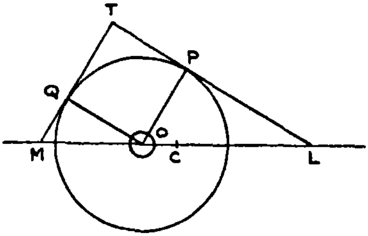
and $(X-a) = r(1+\cos\theta) = a(1-\cos\theta) = 2a\sin^2\frac{\theta}{2}.$

$$\therefore \frac{(X-a)^2}{Y^2} = \operatorname{cosec}^2\frac{\theta}{2}-1 = \frac{2a}{X-a}-1 = \frac{3a-X}{X-a}.$$

$$\therefore (X-a)^3 = Y^2(3a-X)$$

is the equation of the locus of the point of contact required.

Ex. 11. A triangle is formed by the axis of x and two perpendicular tangents to the circle $x^2+y^2 = a^2$. Find the equation to the locus of its nine-point centre. [L.]



As in the diagram, let TL, TM be two perpendicular tangents to the circle $x^2+y^2 = a^2$ whose equations are

$$x\cos\theta+y\sin\theta = a, \qquad -x\sin\theta+y\cos\theta = a,$$

and let them cut the x-axis in L and M. Let C be the middle point of LM.

Then the nine-point centre is N, the middle point of CT, the line joining the circum- and ortho-centres of the triangle TLM.

Now $2OC = OL-OM = a\sec\theta-a\operatorname{cosec}\theta.$

And the coordinates of T are $a(\cos\theta-\sin\theta)$, $a(\cos\theta+\sin\theta)$.

Hence, if (X, Y) be the coordinates of N,

$$X = \tfrac{1}{2}a\left(\cos\theta - \sin\theta + \frac{\sec\theta - \operatorname{cosec}\theta}{2}\right)$$

$$= \tfrac{1}{2}a(\cos\theta - \sin\theta)\left(1 - \frac{1}{\sin 2\theta}\right),$$

$$Y = \tfrac{1}{2}a(\cos\theta + \sin\theta).$$

$$\therefore\ \frac{4Y^2}{a^2} = (\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta.$$

$$\therefore\ 1 - \frac{1}{\sin 2\theta} = \frac{2(a^2 - 2Y^2)}{a^2 - 4Y^2}.$$

$$\therefore\ X.\frac{a^2 - 4Y^2}{a^2 - 2Y^2} = a(\cos\theta - \sin\theta)$$

and $$2Y = a(\cos\theta + \sin\theta).$$

Squaring and adding

$$\frac{X^2(a^2 - 4Y^2)^2}{(a^2 - 2Y^2)^2} + 4Y^2 = 2a^2.$$

$\therefore$ the equation of the required locus is

$$X^2(a^2 - 4Y^2)^2 = 2(a^2 - 2Y^2)^3.$$

✓ Ex. 12. Show that the common chord of two equal circles, one from the system $x^2 + y^2 - 2\kappa x + \delta^2 = 0$ and the other from the system $x^2 + y^2 - 2\lambda y - \delta^2 = 0$, touches the curve whose equation is $2(x^2 - y^2) = \delta^2$.

The circles are equal. $\therefore\ \kappa^2 - \delta^2 = \lambda^2 + \delta^2$.

The equation of their radical axis, found by subtraction of the terms of the two equations, is

$$\kappa x - \lambda y = \delta^2.$$

$$\therefore\ (\kappa x - \delta^2)^2 = \lambda^2 y^2 = (\kappa^2 - 2\delta^2)y^2.$$

$$\therefore\ \kappa^2(x^2 - y^2) - 2\kappa x\delta^2 + \delta^4 + 2\delta^2 y^2 = 0.$$

We now have to express the condition that this equation in κ should have equal roots, which is that

$$x^2\delta^4 = (x^2 - y^2)(\delta^4 + 2\delta^2 y^2),$$

which reduces to $$2(x^2 - y^2) = \delta^2.$$

And this is the equation of a curve which will always be touched by the common chord of the two circles, one from each system.

EXAMPLES 9

1. Find the equations of the tangents to the following circles and also their lengths:

(1) $x^2+y^2 = 9$ from the point $(-6, -7)$.
(2) $x^2+y^2-2\kappa x+\delta^2 = 0$ from the point $(0,\lambda)$.
(3) $x^2+y^2-4x-6y+3 = 0$ from the point $(-5, 8)$.
(4) $x^2+y^2+4x-6y-51 = 0$ from the point $(9, 11)$.

2. If the circle $x^2+y^2+\lambda x+3y-5 = 0$ cuts the circle

$$x^2+y^2+5x+\lambda y+7 = 0$$

orthogonally, find λ.

3. Show that the circles $x^2+y^2-2ax-2by+c^2 = 0$ and $x^2+y^2 = c^2$ cut orthogonally.

4. If the circles $x^2+y^2+2ax+2by = 0$ and $x^2+y^2+2cx+2dy = 0$ are orthogonal, show that $ac+bd = 0$. Also that if $a = 0$, then $d = 0$.

5. Find the equation of the circle which cuts the circles $x^2+y^2 = a^2$ and $(x-b)^2+y^2 = a^2$ orthogonally and passes through the point (b, b).

6. Find the locus of the centre of the circle which cuts two given circles orthogonally.

7. Find the equation of the circle through the point $(1, -2)$ which cuts orthogonally the circles $x^2+y^2-5x-3y+6 = 0$ and $x^2+y^2+7x-y-3 = 0$.

8. Find the radius of the circle whose centre is the point $(-4, -3)$ which cuts orthogonally the circle $(x-3)^2+(y-7)^2 = 5$.

9. Show that the two circles which pass through the points $(0, 3)$, $(0, -3)$ and touch the line $y = \sqrt{7}x+9$ cut each other orthogonally.

10. Find the equation of the circle which cuts orthogonally the three circles whose equations are $x^2+y^2-7x+8y+3 = 0$, $x^2+y^2-4x-3y-8 = 0$, and $x^2+y^2-x-2y+11 = 0$.

11. Show that any circle with its centre on the y-axis will cut orthogonally the circles

$$x^2+y^2+2gx+c = 0, \qquad x^2+y^2+2g'x+c = 0.$$

12. Show that the circles

$$x^2+y^2+ax+by+\tfrac{1}{2} = 0 \quad \text{and} \quad x^2+y^2+\frac{1}{a}x+\frac{1}{b}y+\tfrac{1}{2} = 0$$

are orthogonal for any values of a and b.

13. Show that the lengths of the tangents from the point $(5, 4)$ to the circles $x^2+y^2-4x-7y+6 = 0$ and $x^2+y^2+3x-14y-1 = 0$ are equal. What do you infer as to the position of the point?

14. What is the equation of the radical axis of the circles $2x^2+2y^2-4x-7y+11 = 0$ and $3x^2+3y^2-7x-9y+7\tfrac{1}{2} = 0$? Also find the coordinates of the foot of the radical axis.

15. Show that the circles $x^2+y^2-\kappa(x+y)+\delta^2 = 0$ represent a coaxal system for various values of κ.

16. Find the equation of the circle which passes through the point $(16, 18)$ and through the points of intersection of the circles $x^2+y^2-5x-2y+8 = 0$, $x^2+y^2-3x-y+6 = 0$.

17. What are the limiting points of the circles whose equations are $(x+4)^2+y^2 = 7$, $(x-8)^2+y^2 = 55$?

18. Find the equations of the circles coaxal with the circles $x^2+y^2-6x-1 = 0$ and $x^2+y^2-9x-1 = 0$ which touch the line $x+\sqrt{3}y-2 = 0$.

19. What is the equation of the circle through the origin coaxal with the two circles

$$x^2+y^2-7x-5y-44 = 0 \quad \text{and} \quad x^2+y^2+4x-9y-11 = 0?$$

20. Find the equations of the circles coaxal with the system $x^2+y^2-2\kappa x+\delta^2 = 0$ which touch the line $3x+4y = 5\delta$.

21. If a system of circles are such that a given point has the same polar with regard to any one of them, show that they are coaxal and find the equation of their radical axis.

22. Show that the point of intersection of the polars of a point on the radical axis of the circles $x^2+y^2-2\kappa x+\delta^2 = 0$ and $x^2+y^2-2\kappa x+\delta'^2 = 0$ lies on their radical axis.

23. Find the locus of the middle points of chords through the foot of the radical axis of a coaxal system of circles which intercept a constant length l on lines drawn through this point.

24. Find the locus of the points of contact of parallel tangents to circles of a coaxal system.

25. Show that the polar of a point (h, k) with regard to any one of a coaxal system of circles whose equation is $x^2+y^2-2\kappa x+\delta^2 = 0$ passes through the fixed point $\left(-h, \dfrac{h^2-\delta^2}{k}\right)$.

26. Two circles intersect in points A and B. A line through A meets one circle at P, and a parallel line through B meets the other circle at Q. Show that the locus of the middle point of PQ is a circle.

27. Find the equation of the circle through the points of intersection of the circles

$$x^2+y^2-2\kappa x+\delta^2 = 0 \quad \text{and} \quad x^2+y^2-2\lambda y-\delta^2 = 0$$

which has its centre on the line $y = x$.

28. Show that the circle through the points of intersection of the circles $x^2+y^2-2\kappa x+\delta^2 = 0$ and $x^2+y^2-2\lambda y-\delta^2 = 0$ and the origin passes through the centres of the two circles.

29. Find the coordinates of the centres of similitude of the circles whose equations are: (1) $(x-3)^2+(y+4)^2 = 25$ and $x^2+y^2 = 16$; (2) $x^2+y^2-2\kappa x+\delta^2 = 0$ and $x^2+y^2-2\lambda y-\delta^2 = 0$.

30. Find the equations of the common tangents to the circles $x^2+y^2 = 16$ and $(x-5)^2+(y-6)^2 = 1$.

31. Show that the circle of similitude of the two circles $x^2+y^2 = 16$ and $(x-5)^2+(y-6)^2 = 1$ is coaxal with them.

32. Prove that the equation of the real common tangents to the circles $x^2+y^2 = 2ax$ and $x^2+y^2 = 2by$ is

$$2ab(x^2+y^2-2ax) = (by-ax+ab)^2.$$

33. Find the locus of a point such that the tangents from it to $x^2+y^2 = 3$ and $x^2+y^2 = 3x+6$ are of lengths whose ratio is 2 : 3.

34. Show that the equation of the tangents from the origin to the circle $(x-3)^2+(y+4)^2 = 36$ is $(3y+4x)^2 = 36(x^2+y^2)$.

35. Show that the tangents from the origin to the circle $(x-\alpha)^2+(y-\beta)^2 = r^2$ touch the circle $(x-\lambda\alpha)^2+(y-\lambda\beta)^2 = \lambda^2r^2$.

36. Tangents drawn from the point (h, k) to the circle $x^2+y^2 = a^2$ make angles of α and β with the x-axis. Prove that

$$(1)\ \tan\alpha+\tan\beta = \frac{2hk}{h^2-a^2}, \qquad (2)\ \tan\alpha\tan\beta = \frac{k^2-a^2}{h^2-a^2}.$$

37. A variable circle cuts two given circles at constant angles α and β. If r_1 and r_2 are their radii and d the distance between their centres, prove that the circle cuts the radical axis of the two given circles at an angle θ, where $d\cos\theta = r_1\cos\alpha-r_2\cos\beta$.

Deduce that the length of the perpendicular from the centre of the variable circle on the radical axis of the given circles varies as its radius.

38. Find the equation of the chord joining the points

$$(2a\cos\alpha, \alpha), \qquad (2a\cos\beta, \beta)$$

on the circle whose polar equation is $r = 2a\cos\theta$.

39. Show that, for all values of α, the straight line

$$r\cos(\theta-\alpha) = \rho\cos\alpha+a$$

is a tangent to the circle $r^2-2r\rho\cos\theta+\rho^2-a^2 = 0$.

40. Find the polar coordinates of the points of intersection of the line $r\cos\theta = a$ and the circle $r = 2a\cos\theta$.

41. Show that the equation $r = 2a\cos\theta+2b\sin\theta$ represents a circle and find the length of its radius.

42. Find the condition that the line $\dfrac{1}{r} = a\cos\theta + b\sin\theta$ may touch the circle $r = 2c\cos\theta$.

43. Show that the circles $r = 2a\cos(\theta-\alpha)$ and $r = 2b\cos(\theta-\beta)$ intersect at an angle of $\alpha-\beta$.

44. Prove that the diameter of the circle passing through the pole and the points (r_1, θ_1), (r_2, θ_2) is of length

$$\sqrt{\{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)\}}/\sin(\theta_1-\theta_2).$$

45. Two circles intersect in a point O, and a line through O cuts them in P and Q respectively. Find (1) the locus of the middle point of PQ, (2) the locus of a point which divides PQ in the ratio $\lambda : 1$.

46. O is a fixed point and P moves on a fixed circle. Find the locus of a point Q which divides OP in the ratio $k : l$.

47. Find the radical axis of the circles whose equations are $(x-a)^2+(y-b)^2 = c^2$ and $(x-b)^2+(y-a)^2 = c^2$, and determine c, if the circles touch each other.

48. The polars of a point P with respect to two circles meet in C. Prove that their radical axis bisects PC.

49. If tangents TP, TQ be drawn to two circles from a point T, and TM be drawn perpendicular to the radical axis, show that $TP^2-TQ^2 = \kappa TM$, where κ is a constant.

50. One vertex of a rectangle is a fixed point (h, k), two other vertices lie on a fixed circle $x^2+y^2 = a^2$; show that the locus of the other vertex is the circle $x^2+y^2 = 2a^2-h^2-k^2$.

51. Two points A and B on a circle subtend a right angle at a fixed point O. Find (1) the locus of the middle point of AB, (2) the locus of the pole of AB with regard to the fixed circle.

52. Find the coordinates of the pole of the radical axis with regard to any one of the system of circles whose equation is $x^2+y^2-2\kappa x+\delta^2 = 0$.

53. Find the equation of the tangents from the origin to the circle whose equation is $x^2+y^2+2gx+2fy+c = 0$. Hence show that if these tangents are perpendicular to each other the locus of the centres of the circles obtained by varying g and f would be $x^2+y^2 = 2c$.

54. Show that the points of contact of the external common tangents of the two circles $(x-a)^2+y^2 = r_1^2$ and $(x+a)^2+y^2 = r_2^2$ lie on the circle whose equation is $x^2+y^2 = a^2+r_1 r_2$. [O. and C.]

55. Show that the locus of the point of intersection of two tangents to a circle which contain a constant angle is a concentric circle.

56. The radii of two circles are a and b, and the distance between their centres is $\sqrt{\{2(a^2+b^2)\}}$; prove that a common tangent to the circles subtends a right angle at the point which bisects the distance between their centres.

57. If two circles of a coaxal system touch each other, prove that the product of their radii is constant.

58. From any point on a given circle tangents are drawn to another given circle. Prove that the locus of the middle points of the chords of contact is a circle.

59. ABC is a triangle whose orthocentre is H; prove that the circles on AH and BC as diameters cut orthogonally.

60. Prove that the nine-point circle of the triangle whose vertices are the points $\left(2a, \frac{2}{a}\right)$, $\left(2b, \frac{2}{b}\right)$, $\left(2c, \frac{2}{c}\right)$ passes through the origin.

Formulae of Chapter IV for reference

1. In the general equation of the circle $x^2+y^2+2gx+2fy+c=0$:
Coordinates of centre $(-g, -f)$.
Radius $\sqrt{(g^2+f^2-c)}$.
2. Equation of circle on the line joining (x_1, y_1), (x_2, y_2) as diameter: $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.
3. Equation of chord joining points θ, ϕ:
$$x\cos\frac{\theta+\varphi}{2}+y\sin\frac{\theta+\varphi}{2}=a\cos\frac{\theta-\varphi}{2}.$$
4. Forms of the tangent equations: $xx'+yy'=a^2$.
$x\cos\theta+y\sin\theta=a$.
$y=mx\pm a\sqrt{(1+m^2)}$.
$xx'+yy'+g(x+x')+f(y+y')+c=0$.
5. Equations of the normal: $y=x\tan\theta$; $\dfrac{x}{x'}=\dfrac{y}{y'}$.
6. Equations of the polar of (h, k): $xh+yk=a^2$.
$xh+yk+g(x+h)+f(y+k)+c=0$.
7. Orthogonal condition: $2Gg+2Ff-C-c=0$.
8. Length of tangent from (h, k): $\sqrt{(h^2+k^2-a^2)}$.
$\sqrt{(h^2+k^2+2gh+2fk+c)}$.
9. Standard equation of coaxal circles: $x^2+y^2-2\kappa x+\delta^2=0$.
10. Coordinates of limiting points: $\pm\delta, 0$.
11. Equation of tangents from (h, k):
$(xh+yk-a^2)^2=(x^2+y^2-a^2)(h^2+k^2-a^2)$.
12. Polar equation of a circle: $r=2a\cos\theta$.

V

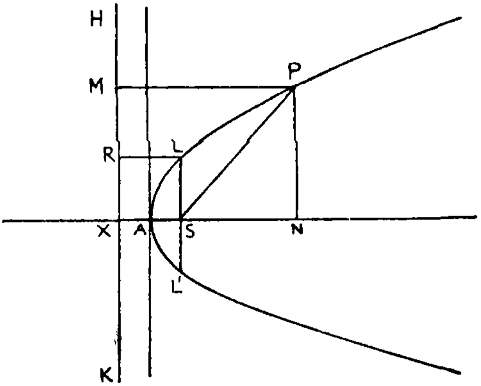
THE PARABOLA

1. Equation of the parabola

As pointed out in the Introduction, if a plane is drawn parallel to a generator of a cone, the curve of its intersection with the conical surface is a *Parabola.*

A *Parabola* is defined as the locus of a point which moves so that its distance from a fixed point, called the *Focus*, is equal to its perpendicular distance from a fixed straight line called the *Directrix.*

To find the equation of a parabola.



Let S be the focus of a parabola and HK its directrix. From S draw SX perpendicular to the directrix HK. Bisect SX at A. Let $SA = AX = a$.

Then, since the point A satisfies the condition of the locus, it is a point on the parabola.

Take any other point P which satisfies the condition of the locus. Draw PM perpendicular to the directrix HK and PN perpendicular to XS or XS produced.

Take the line $XASN$ as x-axis and a line through A perpendicular to it as y-axis.

Let the coordinates of P with regard to these axes be (x, y).

Then by definition $\quad SP = PM$

and $\quad PM = NX. \quad \therefore \; SP^2 = NX^2.$

$$\therefore \; PN^2 = SP^2 - SN^2 = NX^2 - SN^2 = (NX - SN)(NX + SN)$$
$$= SX.2AN = 4AX.AN.$$
$$\therefore \; \mathbf{y^2 = 4ax}.$$

And this is the equation of the parabola referred to the axes selected.

In writing out work, for brevity, the symbol ∈ is often written for 'parabola', just as ○ is for circle.

The point A midway between the focus and the directrix is called the *Vertex* of the parabola.

The line through the vertex and focus of the parabola, which is perpendicular to the directrix, is called the *Axis* of the parabola.

We observe that, if in the equation $y^2 = 4ax$, $-y$ is substituted for y, the form of the equation is unchanged. Hence the parabola is symmetrical with regard to the x-axis which is the axis of the curve.

Further, unless x be positive, there can be no real value of y. This indicates that no part of the curve lies to the left of the y-axis.

Lastly, there is no limit to the positive value of x, and therefore none to that of y. Hence the curve goes off to infinity in two directions, one branch of it being above and the other below its axis.

If through the focus S a line LSL' be drawn perpendicular to the axis and meeting the parabola in L and L', this line LSL' is called the *Latus Rectum* of the parabola and LS or SL' the semi-latus rectum.

Draw LR perpendicular to the directrix.

Then, by definition,
$$SL = LR$$
$$= SX$$
$$= 2a.$$

$\therefore$ $4a$ is the length of the latus rectum of the parabola whose equation is $y^2 = 4ax$.

2. The general equation of a parabola

Suppose that (x, y) are the coordinates of any point on a parabola whose focus is the point (h, k) and directrix the line whose equation is $x\cos\alpha + y\sin\alpha = p$.

Then the square of the distance of (x, y) from the focus (h, k) is
$$(x-h)^2+(y-k)^2.$$

And the square of the length of the perpendicular from (x, y) on the directrix whose equation is $x\cos\alpha+y\sin\alpha = p$ is
$$(x\cos\alpha + y\sin\alpha - p)^2.$$

But, by definition, these are equal.
$$\therefore\ (x-h)^2+(y-k)^2 = (x\cos\alpha + y\sin\alpha-p)^2,$$
i.e.
$$x^2\sin^2\alpha-2xy\sin\alpha\cos\alpha+y^2\cos^2\alpha-2xh-2yk+h^2+k^2-p^2 = 0,$$
which, as h, k, and p are constants, can also be written
$$(x\sin\alpha - y\cos\alpha)^2+2gx+2fy+c = 0.$$

From which the following characteristics should be noted:

(1) *The terms of second degree in x and y form a perfect square.*

(2) The second-degree terms in x and y are the square of a linear expression which, equated to zero, represents a line through the origin perpendicular to the directrix
$$x\cos\alpha + y\sin\alpha - p = 0,$$
and therefore parallel to the axis of the parabola.

Conversely, if in the general equation the terms of the second degree in x and y form a perfect square, we know that the equation represents a parabola.

The general equation of a parabola is usually written

$$(ax+by)^2+2gx+2fy+c = 0.$$

3. Parametric coordinates

The values $x = at^2$, $y = 2at$ clearly satisfy the equation of the parabola $y^2 = 4ax$, no matter what value t may have.

These are, then, *Parametric Coordinates* of any point on this parabola. t is the *Parameter*, i.e. the single variable in terms of which the coordinates can be both expressed. For brevity, we shall speak of 'a point t on a parabola' to indicate that point on it whose coordinates are $(at^2, 2at)$.

4. Equations of a chord and tangent

Let P, Q be two points t and t' on a parabola whose equation is $y^2 = 4ax$.

The equation of the chord PQ is

$$\frac{x-at^2}{at^2-at'^2} = \frac{y-2at}{2at-2at'}.$$

As $t \neq t'$, this becomes

$$2(x-at^2) = (t+t')(y-2at),$$

which can be written

$$x-\frac{t+t'}{2}y+att' = 0. \quad \checkmark$$

To obtain the equation of the tangent at P, we imagine Q to move up to P, i.e. algebraically we put $t' = t$, and get the equation

$$x-ty+at^2 = 0,$$

which is the equation of the tangent at the point t.

The vertex A of the parabola is the point $(0, 0)$, i.e. at A $t = 0$. $\therefore$ the equation of the *tangent at the vertex* is $x = 0$, i.e. it is the y-axis and perpendicular to the axis of the parabola.

Suppose that the coordinates of P and Q instead of being parametrical were (x', y'), (x'', y'').

Then the equation of PQ is

$$\frac{y-y'}{y'-y''} = \frac{x-x'}{x'-x''} = \frac{x-x'}{\dfrac{y'^2}{4a}-\dfrac{y''^2}{4a}},$$

since $\qquad y'^2 = 4ax' \quad\text{and}\quad y''^2 = 4ax''.$

So the equation becomes, as $y' \neq y''$,

$$(y-y')\frac{y'+y''}{4a} = x-x' = x-\frac{y'^2}{4a},$$

i.e. $$x-y\frac{y'+y''}{4a}+\frac{y'y''}{4a} = 0.$$

The tangent at the point (x',y'), obtained by putting $y' = y''$ in this equation, is

$$yy' = 2a\left(x+\frac{y'^2}{4a}\right)$$

or $$\mathbf{yy' = 2a(x+x').}$$

The equation of the tangent at the point (x',y') to the parabola $y^2 = 4ax$ may also be obtained very simply by the use of the calculus, if the student is familiar with this. Thus, differentiating the equation $y^2 = 4ax$ with regard to x, we have

$$2y\frac{dy}{dx} = 4a, \quad \text{whence } \frac{dy}{dx} = \frac{2a}{y}.$$

The equation of the tangent at (x',y') is accordingly

$$y-y' = \frac{2a}{y'}(x-x'),$$

i.e. $$yy' = 2ax+y'^2-2ax' = 2ax+2ax'$$
$$= 2a(x+x').$$

Both forms of the equation of the tangent, parametric and cartesian, are useful. Nevertheless parametric coordinates are ordinarily to be preferred, involving, as they do, only a single variable as against two in the case of cartesians. More especially is this the case where, in finding the equation of a locus, the process involves the elimination of only one variable when parametric coordinates are employed.

5. Equation of the normal

To find the equation of the normal to a parabola whose equation is $y^2 = 4ax$ at the point t, all that is necessary is to find the equation of the line through the point whose coordinates are $(at^2, 2at)$ perpendicular to the tangent whose equation is $x - ty + at^2 = 0$.

The required equation is clearly

$$tx + y = t(at^2) + 2at,$$

i.e.

$$\mathbf{tx + y = at^3 + 2at.}$$

Similarly, the equation of a line through (x', y') perpendicular to the tangent whose equation is $yy' = 2a(x + x')$ is

$$xy' + 2ay = x'y' + 2ay'$$

or

$$y'(x - x') + 2a(y - y') = 0.$$

6. Two tangents to a parabola

The equation of the tangent at any point t on the parabola $y^2 = 4ax$ can be written $at^2 - ty + x = 0$.

Suppose that this passes through a fixed point (h, k), then

$$at^2 - tk + h = 0,$$

which is a quadratic in t. Suppose that its roots are t_1 and t_2. Then it follows that for any fixed point (h, k) there will be two tangents passing through it which will touch the parabola at the points t_1 and t_2. These tangents will be real if t_1 and t_2 are, i.e. if $k^2 > 4ah$. In this case (h, k) must lie outside the parabola.

They will be coincident if $k^2 = 4ah$, i.e. if (h, k) lies on the parabola.

They will be imaginary if $k^2 < 4ah$, i.e. if (h, k) lies within the parabola.

Here it may also be observed that $t_1+t_2 = \frac{k}{a}$ and $t_1 t_2 = \frac{h}{a}$.

Hence the point where the tangents at two points t_1 and t_2 meet has for its coordinates $\mathbf{at_1 t_2}$ and $\mathbf{a(t_1+t_2)}$.

7. The tangential equation of a parabola

To find the tangential equation is equivalent to finding the condition that the line $lx+my+n = 0$ should touch the parabola whose equation is $y^2 = 4ax$.

Identify $lx+my+n = 0$ with the equation of the tangent at any point t, viz. with $x-ty+at^2 = 0$. We then have

$$\frac{1}{l} = -\frac{t}{m} = \frac{at^2}{n}.$$

$$\therefore \frac{m^2}{l^2} = t^2 = \frac{n}{la},$$

i.e. $$am^2 = ln.$$

Identifying the line whose equation is $y = mx+c$ with the equation of the tangent at any point t, viz. with $x-ty+at^2 = 0$ or $y = \frac{x}{t}+at$, we have

$$m = \frac{1}{t} \quad \text{and} \quad c = at = \frac{a}{m}.$$

$\therefore$ the line $y = mx+\frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ for any value of m.

8. Pole and polar with regard to a parabola

The point of intersection of two tangents to a parabola is called the *Pole* of the line which joins their points of contact, and this chord is called the *Polar* of the point, with regard to the parabola.

As explained in the case of the circle, if the point be within the parabola the tangents are imaginary and so accordingly are their points of contact; nevertheless the polar of the point is always a real line. Its equation can be obtained as follows:

It has been proved that the tangents to a parabola at two points t_1, t_2 meet in a point whose coordinates are at_1t_2, $a(t_1+t_2)$.

Suppose this is the point (h, k), where

$$h = at_1t_2 \quad \text{and} \quad k = a(t_1+t_2).$$

The polar of the point is, by definition, the chord joining the points t_1, t_2 whose equation is

$$x-\frac{t_1+t_2}{2}y+at_1t_2 = 0,$$

i.e.

$$x-\frac{k}{2a}y+h = 0$$

or

$$\boxed{yk = 2a(x+h).}$$

This equation, we note, is the same in form as that of the tangent to the parabola, and if (h, k) were on the parabola it would be the equation of the tangent at this point on it.

Hence the polar of a point (h, k) is always a real line, since its equation contains only real quantities, and the polar of a point on the parabola is the tangent at that point.

As in the case of the circle, it can be proved that if the polar of a point (h, k) with regard to a parabola whose equation is $y^2 = 4ax$ passes through another fixed point (H, K), then the polar of (H, K) will pass through (h, k).

The proof is briefly as follows:

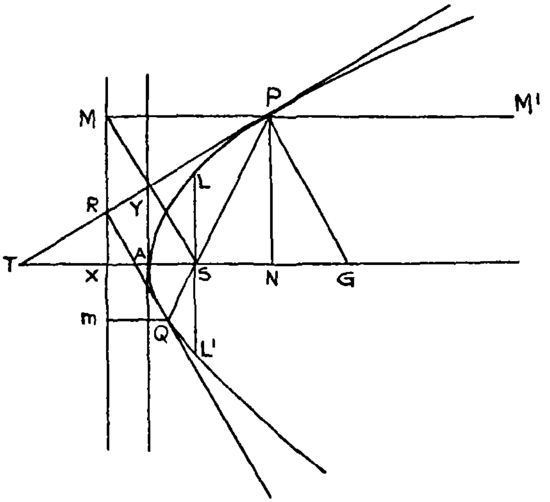
The polar of (h, k) with regard to the parabola is

$$yk = 2a(x+h).$$

If this passes through the point (H, K), then $Kk = 2a(H+h)$. But the polar of (H, K) is $yK = 2a(x+H)$.

The last two equations show that $x = h$, $y = k$ satisfies the equation of the polar of (H, K), i.e. the polar of (H, K) passes through the point (h, k).

9. Various properties of the parabola



In the above diagram S is the focus of a parabola, XS is its axis, and XM its directrix.

P is any point t on the parabola. Through P is drawn PN perpendicular to the axis and PM perpendicular to the directrix. MP is produced to M'.

LSL' is the latus rectum. PSQ is a focal chord. The tangents at P and Q meet in R.

The tangent and normal at P meet the axis in T and G respectively. The tangent at the vertex A meets PT in Y.

Qm is drawn perpendicular to the directrix.

Join SY, YM.

We shall now proceed to prove several important theorems connected with the parabola whose equation is $y^2 = 4ax$ referred to its axis and the tangent at its vertex as axes of coordinates. They will be numbered in brackets as we proceed.

(1) $AT = AN$.

The tangent at P, the point t, is $x-ty+at^2 = 0$.

Where this cuts $y = 0$, we have $x = -at^2$.

$\therefore$ AT and AN are both of length at^2 and so equal.

NT is often called the *Subtangent* of the point P, and its length, as just proved, is $2at^2$.

(2) $SP = ST$ and $\angle SPT = \angle STP$.

From the diagram

$$SP = PM = NX = AX+AN = a+at^2$$

and $$ST = SA+AT = a+at^2;$$

whence $SP = ST$ and $\angle SPT = \angle STP$.

(3) PT bisects $\angle SPM$ and PG bisects $\angle SPM'$.

From the diagram

$$\begin{aligned}\angle MPT &= \text{alternate } \angle STP \\ &= \angle SPT \text{ from (2).}\end{aligned}$$

$\therefore$ PT bisects $\angle SPM$.

Also the normal PG, being perpendicular to PT, must bisect the exterior $\angle SPM'$.

(4) The tangents at the extremities of a focal chord meet at right angles on the directrix.

Let PQ be the focal chord, P the point t, Q the point t'.

The equation of PQ is $x-\frac{1}{2}(t+t')y+att' = 0$.

Since this passes through S the point $(a, 0)$, we have $a+att' = 0$. $\therefore$ $t' = -\dfrac{1}{t}$.

Now the tangents at the points t and t' meet at a point whose abscissa is $att' = -a$.

$\therefore$ R the point where the tangents at P and Q meet is on the directrix.

Also, the gradients of the tangents at t and t' having their product $\dfrac{1}{t}\cdot\dfrac{1}{t'}$ equal to -1 are perpendicular. Hence the tangents at P and Q meet at right angles at a point R on the directrix of the parabola.

(5) To prove that RS is perpendicular to PQ.

As in (4), R is the point $-a, a\left(t-\frac{1}{t}\right)$ and S the point $(a, 0)$.

$\therefore$ the equation of RS is

$$\frac{x+a}{-a-a} = \frac{y-a\left(t-\frac{1}{t}\right)}{a\left(t-\frac{1}{t}\right)},$$

of which line the gradient is $-\frac{1}{2}\left(t-\frac{1}{t}\right)$.

But the gradient of PQ, the line $x-\frac{1}{2}\left(t-\frac{1}{t}\right)y-a = 0$, is

$$\frac{2}{t-\frac{1}{t}}.$$

$\therefore$ the product of the gradients of RS and PQ is -1.

$\therefore$ RS is perpendicular to PQ.

(6) To show that S, Y, and M are collinear, that $PY = YT$, that $SY = YM$, and that SY is perpendicular to PT.

Where the tangent at P, $x-ty+at^2 = 0$, cuts the tangent at the vertex $x = 0$, we have $y = at$.

$\therefore$ $AY = \frac{1}{2}PN$ and $\frac{AY}{AS} = \frac{PN}{2AS} = \frac{MX}{XS}$.

$\therefore$ S, Y, and M are collinear.

Again, as $AY = \frac{1}{2}PN$ and AY and PN are parallel,

$$\therefore \; PY = YT.$$

In the triangles SPY, MPY, $SP = PM$, PY is common, and $\angle SPY = \angle MPY$. $\therefore$ $SY = YM$.

Also, $\angle SYP = \angle NYP$ and each is a right angle.

$\therefore$ SY is perpendicular to PT.

Hence, *the foot of the perpendicular from the focus on any tangent always lies on the tangent at the vertex.*

(7) To show that $SY^2 = SA.SP$.

$$\frac{SY}{SP} = \sin\angle SPT = \sin\angle STY = \cos\angle ASY = \frac{SA}{SY}.$$

$$\therefore\ SY^2 = SA.SP.$$

(8) If the normal at a point P cuts the axis in G, then $SG = SP$; also NG (the *subnormal* of the point P) $= 2a$.

The equation of the normal at P $(at^2, 2at)$ is

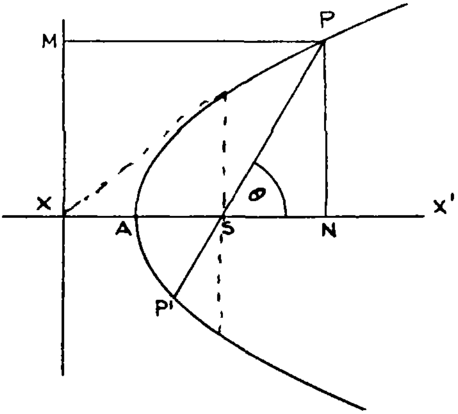
$$tx+y = at^3+2at.$$

Where PG cuts the axis $y = 0$, $x = at^2+2a$.

$$\therefore\ SG = AG-AS = at^2+2a-a = a(1+t^2) = SP$$

and $$NG = AG-AN = at^2+2a-at^2 = 2a.$$

10. Polar equation of a parabola



Take S as the pole and XAS as the initial line.

Let P be any point (r, θ) on the parabola, θ being reckoned in an anti-clockwise direction.

Draw PN perpendicular to the axis XAS, produced if necessary. Produce PS to cut the curve again in P'. Draw PM perpendicular to the directrix XM.

Then

$$r = SP = PM = NX = XS+SN = 2a+SP\cos\theta$$
$$= l+r\cos\theta,$$

where l is the semi-latus rectum of the parabola.

Hence $\boldsymbol{\frac{l}{r} = 1-\cos\theta}$ is the polar equation of the parabola.

Sometimes θ is reckoned in a clockwise direction from SX. In this case

$$\frac{l}{r} = 1+\cos\theta$$

would be the polar equation of the parabola.

The polar equation of the directrix is clearly $-r\cos\theta = SX$,

i e. $$r\cos\theta+l = 0.$$

(1) To prove that $\frac{1}{SP}+\frac{1}{SP'} = \frac{2}{l}$.

From the polar equation of the parabola $\frac{l}{r} = 1-\cos\theta$,

$$\frac{l}{SP} = 1-\cos\theta,$$

also $$\frac{l}{SP'} = 1-\cos(\pi+\theta) = 1+\cos\theta.$$

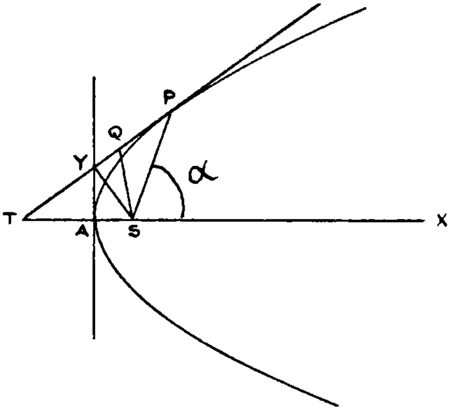
$$\therefore\ \frac{l}{SP}+\frac{l}{SP'} = 2,$$

i.e. $$\frac{1}{SP}+\frac{1}{SP'} = \frac{2}{l}.$$

To find the polar equation of the tangent at a point on the parabola $\frac{l}{r} = 1-\cos\theta$.

Let P be a point whose coordinates are $\left(\frac{l}{1-\cos\alpha}, \alpha\right)$ on the parabola whose equation is $\frac{l}{r} = 1-\cos\theta$.

Let Q be a point on the tangent at P whose coordinates are (r, θ).



Then from the diagram $\angle QSX = \theta$, $\angle PSX = \alpha$, and

$$\angle QSY = \angle PSY - \angle PSQ = \tfrac{1}{2}(\pi-\alpha)-(\theta-\alpha) = \frac{\pi}{2}-\left(\theta-\frac{\alpha}{2}\right),$$

also

$$SQ\cos\angle QSY = SY = a\sec\angle ASY = a\operatorname{cosec}\angle STP.$$

$$\therefore\ r\cos\left\{\frac{\pi}{2}-\left(\theta-\frac{\alpha}{2}\right)\right\} = a\operatorname{cosec}\frac{\alpha}{2},$$

i.e.
$$r\sin\left(\theta-\frac{\alpha}{2}\right) = a\operatorname{cosec}\frac{\alpha}{2}.$$

$$\therefore\ \frac{l}{r}=\frac{2a}{r} = 2\sin\left(\theta-\frac{\alpha}{2}\right)\sin\frac{\alpha}{2} = \cos(\theta-\alpha)-\cos\theta,$$

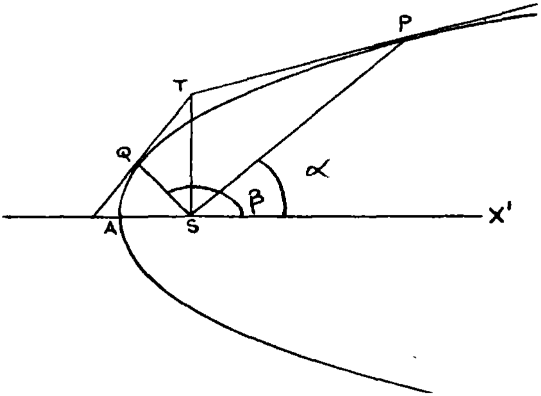
which is the required equation.

(2) If TP, TQ be two tangents to a parabola, to prove that (1) $\angle TSP = \angle TSQ$, (2) $SP.SQ = ST^2$.

Let P and Q be the points $\left(\dfrac{l}{1-\cos\alpha}, \alpha\right)$, $\left(\dfrac{l}{1-\cos\beta}, \beta\right)$.

The polar equations of the tangents at P and Q are

$$\frac{l}{r} = \cos(\theta-\alpha)-\cos\theta \quad\text{and}\quad \frac{l}{r} = \cos(\theta-\beta)-\cos\theta.$$



Where these tangents intersect, we have

$$\cos(\theta-\alpha) = \cos(\theta-\beta),$$

and since θ is not zero, $\therefore\ \theta-\alpha = -(\theta-\beta)$.

$$\therefore\ \theta = \frac{\alpha+\beta}{2}.$$

$$\therefore\ \angle TSP = \frac{\alpha+\beta}{2}-\alpha = \frac{\beta-\alpha}{2}$$

and $$\angle TSQ = \beta-\frac{\alpha+\beta}{2} = \frac{\beta-\alpha}{2}.$$

Hence $$\angle TSP = \angle TSQ.$$

Again, at the point T we have

$$\frac{l}{ST} = \cos\left(\frac{\alpha+\beta}{2}-\alpha\right)-\cos\frac{\alpha+\beta}{2}$$

$$= \cos\frac{\beta-\alpha}{2} - \cos\frac{\alpha+\beta}{2}$$

$$= 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2},$$

but $\quad \dfrac{l}{SP} = 1-\cos\alpha = 2\sin^2\dfrac{\alpha}{2} \quad$ and $\quad \dfrac{l}{SQ} = 2\sin^2\dfrac{\beta}{2}.$

$$\therefore\ SP.SQ = \frac{l^2}{4\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}} = ST^2.$$

From these results it follows that the triangles SPT, SQT are similar.

The polar equation of the normal to a parabola at the point $\left(\dfrac{l}{1-\cos\alpha}, \alpha\right)$ can be found thus:

The equation of the tangent at P is

$$\frac{l}{r} = \cos(\theta-\alpha)-\cos\theta.$$

The equation of any line perpendicular to this is

$$\frac{\kappa}{r} = \cos\left(\theta+\frac{\pi}{2}-\alpha\right)-\cos\left(\theta+\frac{\pi}{2}\right).$$

If this passes through the point $\left(\dfrac{l}{1-\cos\alpha}, \alpha\right)$, we have

$$\frac{\kappa(1-\cos\alpha)}{l} = \sin\alpha.$$

$\therefore$ the equation of the normal is

$$\frac{l\sin\alpha}{1-\cos\alpha}.\frac{1}{r} = \sin\theta-\sin(\theta-\alpha).$$

(3) To prove that, if the tangents at three points P, Q, R of a parabola intersect to form the triangle pqr, the circum-circle of the triangle pqr passes through the focus.

Let SP, SQ, SR make the angles α, β, γ with the axis of the parabola $\frac{l}{r} = 1-\cos\theta$ whose focus is S.

In the previous example it has been proved that the coordinates of the points p, q, r will then be

$$\frac{l}{2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}},\quad \frac{\beta+\gamma}{2};\qquad \frac{l}{2\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}},\quad \frac{\gamma+\alpha}{2};$$

and
$$\frac{l}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}},\quad \frac{\alpha+\beta}{2}.$$

Consider the equation

$$r = \frac{l}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}\sin\left(\frac{\alpha+\beta+\gamma}{2}-\theta\right).$$

It is one of a circle which goes through the pole, and the diameter of it through the pole makes an angle $\frac{\alpha+\beta+\gamma}{2}-\frac{\pi}{2}$ with the central line, the axis of the parabola.

The coordinates of each of the points p, q, r clearly satisfy this equation on substitution of them in it.

$\therefore$ the equation is that of a circle circumscribing the triangle pqr.

And the focus of the parabola, taken as pole, lies on it.

The following examples further illustrate the use of polar coordinates in the case of a parabola.

Ex. 1. Prove that the polar equation of a parabola with regard to the vertex as pole and the axis of the parabola as initial line is

$$r\sin^2\theta = 4a\cos\theta.$$

Let P be any point on the parabola whose coordinates are (r, θ) with regard to the vertex A as pole and the axis AS as initial line.

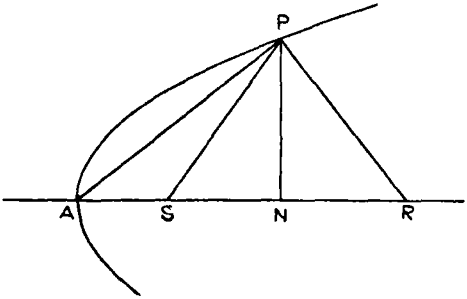
Join AP, SP; through P draw PR perpendicular to AP to cut the axis in R. Also draw PN perpendicular to the axis.

Then, as in the diagram, $PR = AP \tan PAN = r \tan\theta$.

Also $r\tan\theta . \sin\theta = PR\sin\theta = PR\sin RPN = RN$.

But as $\angle APR$ is right, $\therefore\ PN^2 = AN.NR$.

$$\therefore\ RN = \frac{PN^2}{AN} = 4a.$$



$\therefore$ the required equation is

$$r\sin^2\theta = 4a\cos\theta.$$

Ex. 2. The tangents at P and Q to a parabola meet at T. Prove that $TP^2 : TQ^2 = SP : SQ$.

In the diagram (see p. 151) P is the point $\left(\frac{l}{1-\cos\alpha}, \alpha\right)$, Q is $\left(\frac{l}{1-\cos\beta}, \beta\right)$.

We know that $\angle TPS = \frac{\alpha}{2}$ and that

$$\angle TSP = \angle TSQ = \frac{\beta-\alpha}{2}.$$

$$\therefore\ \angle STQ = \angle SPT = \frac{\alpha}{2}.$$

Hence
$$\frac{TP}{TS} = \frac{\sin\angle TSP}{\sin\angle TPS} = \frac{\sin\dfrac{\beta-\alpha}{2}}{\sin\dfrac{\alpha}{2}}$$

and
$$\frac{TQ}{TS} = \frac{\sin\angle TSQ}{\sin\angle TQS} = \frac{\sin\dfrac{\beta-\alpha}{2}}{\sin\dfrac{\beta}{2}}.$$

$$\therefore\ \frac{TP^2}{TQ^2} = \frac{\sin^2\dfrac{\beta}{2}}{\sin^2\dfrac{\alpha}{2}} = \frac{1-\cos\beta}{1-\cos\alpha} = \frac{\dfrac{l}{SQ}}{\dfrac{l}{SP}}$$
$$= \frac{SP}{SQ}.$$

Ex. 3. Find the locus of middle points of focal chords of a parabola.

Let PSQ be a focal chord and H its middle point.

Then
$$SP-SQ = (SH+HP)-(QH-SH)$$
$$= 2SH,$$
since $HP = QH$.

Hence, if PSQ make an angle θ with the axis,
$$2SH = \frac{l}{1-\cos\theta} - \frac{l}{1+\cos\theta} = \frac{2l\cos\theta}{\sin^2\theta}.$$

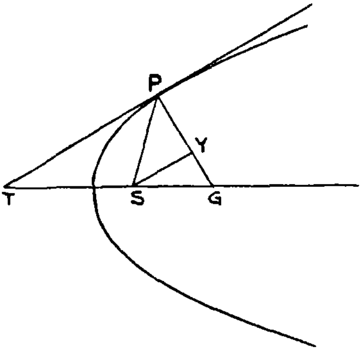
The required locus is therefore
$$r\sin^2\theta = 2a\cos\theta,$$
viz. that of a parabola with S as vertex and latus rectum half that of the original parabola.

Ex. 4. Show that the locus of the foot of the focal perpendicular on the normal at any point of a parabola is another parabola.

Let the perpendicular from S to the normal PG of a parabola cut it in Y.

Let $SY = r$, $\angle YSG = \theta$.

Then, since $SP = SG$, SY bisects $\angle PSG$. $\therefore$ $\angle PSG = 2\theta$.



$$\therefore\ 1-\cos 2\theta = \frac{l}{SP} = \frac{l}{SY\sec\theta}.$$

$$\therefore\ r\sin^2\theta = \frac{l}{2}\cos\theta = a\cos\theta.$$

Hence the locus of Y is a parabola with S as vertex and its latus rectum a quarter that of the original parabola.

11. Typical examples

Ex. 1. To find the coordinates of the vertex and focus and also the length of the latus rectum of the parabola whose equation is $y^2+ax+by+c = 0$.

Rewrite the equation

$$\left(y+\frac{b}{2}\right)^2 = -ax-c+\frac{b^2}{4}$$

$$= -a\left(x-\frac{b^2-4c}{4a}\right).$$

Transfer to parallel axes through the point $\left(\frac{b^2-4c}{4a}, -\frac{b}{2}\right)$. The equation then becomes $y^2 = -ax$.

Its latus rectum is a and the coordinates of its vertex and focus $(0,0)$, $\left(-\frac{a}{4}, 0\right)$.

$\therefore$ the coordinates of the vertex and focus referred to the original axes are $\left(\frac{b^2-4c}{4a}, -\frac{b}{2}\right)$ and $\left(\frac{b^2-4c-a^2}{4a}, -\frac{b}{2}\right)$ respectively.

Ex. 2. If the extremities of a focal chord PSQ be joined to the vertex A of a parabola, show that AP, AQ cut its latus rectum in two points whose distances from the focus are equal to the ordinates of the extremities of the chord.

PSQ is a focal chord. $\therefore$ if P be the point t, Q will be the point $-\frac{1}{t}$.

The equation of AP is $x = \frac{yt}{2}$ and of AQ is $x = -\frac{y}{2t}$. Where these cut $x = a$ we have $y = \frac{2a}{t}$ and $y = -2at$. And the distances of these are the same in length as the ordinates of P and Q.

Ex. 3. Prove that if $t_1 = \lambda t_2$ the locus of the point of intersection of the tangents at the points t_1, t_2 of the parabola $y^2 = 4ax$ is another parabola.

Let (X, Y) be a point on the locus.

Then $X = at_1t_2 = a\lambda t_2^2$ and $Y = a(t_1+t_2) = at_2(\lambda+1)$.

Eliminating t_2, the required locus is

$$\frac{Y^2}{a^2(\lambda+1)^2} = \frac{X}{a\lambda},$$

i.e. the parabola $$Y^2 = \frac{a(\lambda+1)^2}{\lambda}X.$$

Ex. 4. Find the locus of the point of intersection of the tangents at the extremities of any chord of a parabola $y^2 = 4ax$ which subtends a right angle at the vertex.

Let (X, Y) be a point on the locus.

The equation of the chord, i.e. the polar of (X, Y), is

$$yY = 2a(x+X),$$

i.e.
$$-\frac{x}{X}+\frac{yY}{2aX} = 1.$$

The equation of the lines joining the vertex $(0, 0)$ to the intersection of this line and the parabola $y^2 = 4ax$ is

$$y^2 = 4ax\left(-\frac{x}{X}+\frac{yY}{2aX}\right).$$

As these lines are perpendicular, the sum of the coefficients of their x^2 and y^2 terms is zero.

$$\therefore\ 1+\frac{4a}{X} = 0,$$

i.e. the locus is the line $x = -4a$.

Ex. 5. Find the coordinates of the pole of the line $lx+my = 1$ with regard to the parabola $y^2 = 4ax$.

The polar of (h, k) with regard to the parabola is

$$yk = 2a(x+h).$$

Identifying this equation with $lx+my = 1$, we have

$$2ah = \frac{k}{m} = -\frac{2a}{l}.$$

$$\therefore\ h = -\frac{1}{l} \quad\text{and}\quad k = -\frac{2am}{l}.$$

Ex. 6. The orthocentre of the triangle formed by three tangents to a parabola lies on the directrix.

Let the equation of the parabola be $y^2 = 4ax$ and t_1, t_2, t_3 the points of contact of the three tangents.

The vertices of the triangle formed by the three tangents are the points whose coordinates are

$$at_2t_3, a(t_2+t_3);\ at_3t_1, a(t_3+t_1);\ at_1t_2, a(t_1+t_2).$$

The equation of the line through the first of these perpendicular to the tangent at t_1, i.e. to $x-t_1y+at_1^2 = 0$, is

$$t_1x+y = t_1(at_2t_3)+a(t_2+t_3).$$

Where this line cuts the directrix $x = -a$ we have

$$y = at_1t_2t_3+a(t_1+t_2+t_3).$$

The symmetry of this result shows that had we taken either of the other two vertices and formed the equations of the perpendiculars to the opposite sides and then found the points where they cut the directrix $x = -a$, we should have had the same point.

Hence the orthocentre lies on the directrix and is the point $(-a, at_1t_2t_3+at_1+at_2+at_3)$.

Ex. 7. The chord PQ of the parabola $y^2 = 4ax$ passes through the foot of the directrix, the point $(-a, 0)$. Find the locus of the intersection of the normals at P and Q.

Let P and Q be the points t_1, t_2.

The equation of the chord PQ is $x-\frac{t_1+t_2}{2}y+at_1t_2 = 0$. As this goes through the point $(-a, 0)$, we have $t_1t_2 = 1$. The normals at P and Q are

$$t_1x+y = at_1^3+2at_1,$$

$$t_2x+y = at_2^3+2at_2.$$

Solving these equations we get $x = a(t_1^2+t_2^2+t_1t_2+2)$ and $y = -a(t_1+t_2)$.

$$\therefore\ x-3a = a(t_1^2+t_2^2) = a(t_1+t_2)^2-2a = \frac{y^2}{a}-2a.$$

$\therefore$ the equation of the locus is $y^2 = a(x-a)$.

✓Ex. 8. Find the locus of the pole of normal chords of the parabola $y^2 = 4ax$.

Let (X, Y) be a point on the locus.

The polar of (X, Y) is $yY = 2a(x+X)$.

Identifying this with the equation of the normal at any point t, i.e. with

$$tx+y = at^3+2at,$$

we have $$-\frac{2a}{t} = Y = \frac{2aX}{at^3+2at} = \frac{2X}{t^3+2t};$$

whence $$Y\left(-\frac{8a^3}{Y^3}-\frac{4a}{Y}\right) = 2X,$$

i.e. $$Y^2(X+2a)+4a^3 = 0,$$

which is the equation of the required locus.

Ex. 9. Prove that the middle point of the intercept made on a tangent to a parabola by the tangents at two points P and Q lies on the tangent which is parallel to PQ. [O. and C.]

Let the points of contact of two fixed tangents be the points m_1, m_2, and let the point t be the point of contact of a variable tangent.

The coordinates of the points at which the variable tangent cuts the fixed tangents are $atm_1, a(t+m_1)$ and $atm_2, a(t+m_2)$.

If (X, Y) be the middle point of the line joining these two points, $X = \frac{1}{2}at(m_1+m_2)$, $Y = \frac{1}{2}a(m_1+m_2+2t)$.

The equation of the tangent parallel to the chord joining the two points m_1, m_2 is

$$x-\frac{m_1+m_2}{2}y+a\left(\frac{m_1+m_2}{2}\right)^2 = 0.$$

And if the values of X and Y obtained above are substituted for x and y in the left-hand side of this equation, the result is

$$\tfrac{1}{2}at(m_1+m_2)-\tfrac{1}{4}a(m_1+m_2)(m_1+m_2+2t)+\tfrac{1}{4}a(m_1+m_2)^2,$$

which is clearly zero.

$\therefore$ the middle point of the intercept does lie on the tangent parallel to the chord joining the points of contact of the two fixed tangents.

Ex. 10. If the normal at the point P of a parabola makes an angle ψ with the x-axis and meets the parabola again at the point Q, show that $PQ = 4a\sec\psi\operatorname{cosec}^2\psi$. [O. and C.]

Let P be the point t on the parabola $y^2 = 4ax$ and Q the point t_1.

The equation of the chord PQ is $x-\frac{t+t_1}{2}y+at_1t_2 = 0$ and of the normal at P is $tx+y = at^3+2at$.

If these are one and the same, $\frac{2}{t+t_1} = -t = \tan\psi$.

$$\therefore\ t_1 = -\frac{2+t^2}{t}, \qquad t_1-t = -\frac{2(1+t^2)}{t} = 2\sec^2\psi\cot\psi,$$

and $$t_1+t = -\frac{2}{t} = 2\cot\psi.$$

$$\begin{aligned}\therefore\ PQ^2 &= a^2(t_1^2-t^2)^2+4a^2(t_1-t)^2\\ &= a^2(t_1-t)^2\{(t_1+t_2)^2+4\}.\\ \therefore\ PQ &= 2a\sec^2\psi\cot\psi\sqrt{(4\cot^2\psi+4)}\\ &= 4a\sec\psi\,\mathrm{cosec}^2\psi.\end{aligned}$$

Ex. 11. Prove that the locus of the feet of perpendiculars drawn from the origin to the tangents of the parabola $y^2 = 4ax$ is the curve $x(x^2+y^2)+ay^2 = 0$. [W.]

The equation of the tangent at any point t is

$$x-ty+at^2 = 0.$$

The perpendicular through (0, 0) to this line is $tx+y = 0$. *Where these intersect we have*

$$x-y\left(-\frac{y}{x}\right)+a\left(-\frac{y}{x}\right)^2 = 0,$$

i.e. $$x(x^2+y^2)+ay^2 = 0.$$

Ex. 12. A parabola is drawn through four points t_1, t_2, t_3, t_4 of the parabola $y^2 = 4ax$. Show that the axis of the second parabola makes with the axis of the first an angle θ, where $4\cot\theta = t_1+t_2+t_3+t_4$. [J.]

Let the equation of the second parabola be

$$(px+qy)^2+2gx+2fy+c = 0.$$

Its axis is parallel to $px+qy = 0$. $\therefore\ \cot\theta = -\frac{q}{p}$.

In the equation of the second parabola substitute $x = at^2$, $y = 2at$. Then

$$(pat^2+2qat)^2+2gat^2+4fat+c = 0.$$

This is an equation of the fourth degree in t, and its roots must be $t_1+t_2+t_3+t_4$.

$$\therefore\ t_1+t_2+t_3+t_4 = -\frac{\text{coefficient of the } t^3 \text{ term}}{\text{coefficient of the } t^4 \text{ term}} = -\frac{4a^2pq}{p^2a^2}$$

$$= -\frac{4q}{p} = 4\cot\theta.$$

EXAMPLES 10

1. What are the coordinates of the vertex and focus, the length of the latus rectum, and the equation of the directrix of the following parabolas?

(1) $y^2 = 12(x-4)$.
(2) $y^2 = 4a(x+c)$.
(3) $x^2 = 4a(y+c)$.
(4) $(x+3)^2 = 2y-5$.
(5) $y^2+3x-21 = 0$.
(6) $(y-b)^2 = 3a(x+2c)$.
(7) $y^2-4y+2x = 0$.
(8) $y^2-4y+6x-10 = 0$.
(9) $y^2+3x+4y = 27$.
(10) $x^2+6x-7y = 23$.
(11) $4y^2+3x+8y = 15$.
(12) $(3x+4y-2)^2 = 25(x^2+y^2)$.

2. Using only the definition of a parabola, find the equations of the parabolas whose focus and directrix are the following:

(1) $(a, -a)$, $y = 0$.
(2) $(1, -1)$, $3x-4y = 8$.
(3) $(2, 3)$, $5x-12y = 26$.
(4) (p, q), $lx+my = 1$.

3. Find the equation of the parabola which has the point $(2, 3)$ for its focus and the line $3x-4y = 2$ as the tangent at its vertex.

4. Taking the equation of a parabola as $y^2 = 4ax$, what will be the parameters of the points $(0, 0)$, $(a, 2a)$, $(a, -2a)$, $(4a, -4a)$, $\left(\frac{a}{2}, \sqrt{2}a\right)$?

5. Show that the points whose coordinates are

(1) $x = at^2+1$, $y = ct+1$,   (2) $x = at+bt^2$, $y = ct+at^2$,

will in both cases lie on parabolas. Find their equations, t being a variable.

6. Find the coordinates of the points in which the parabola $(2x+y)^2 = 12x-6y-8$ cuts the axes of coordinates.

7. Find the equation of the parabola the coordinates of any point on which are given by

$$x = u\cos\alpha . t, \qquad y = u\sin\alpha . t - \tfrac{1}{2}gt^2,$$

t being a variable.

[N.B. This result is a useful one in dynamics for determining the path of a projectile.]

8. A point moves so that its distance from the x-axis is equal to the length of the tangent drawn from it to the circle

$$x^2+y^2+6x-8y-19 = 0.$$

Show that it traces out a parabola.

9. Find the equation of the parabola whose vertex is the point (h, k), whose latus rectum is $4c$, and whose axis is parallel to that of the parabola $y^2 = 4ax$.

10. Find the equations of the tangent and normal

(1) at the point $(16, -4)$ on the parabola $y^2 = x$.
(2) at the point $(3, -6)$ on the parabola $y^2 = 12x$.
(3) at the point $(6, 3)$ on the parabola $x^2 = 12y$.
(4) at the point $(7, 6)$ on the parabola $(y-2)^2 = 4(x-3)$.
(5) at the point $(10, 4)$ on the parabola $(x-4)^2 = 6(y+2)$.
(6) at the point (x', y') on the parabola $y^2+ax+by = 0$.

11. Find the equations of the tangents and normals at the extremities of the latus rectum of the parabola $y^2 = 4ax$.

12. For what values of c will the lines

$$(1)\ 3y = 2x+c, \qquad (2)\ x-y = c$$

be tangents to the parabola $y^2 = 4ax$?

13. Show that the line $y = 3x+9\frac{5}{12}$ is a tangent to the parabola $y^2 = 5x+15$.

14. Prove that the straight line $lx+my+n = 0$ is a tangent to the parabola $y^2 = 4a(x-c)$, if $l^2c+ln-am^2 = 0$.

15. Prove that the tangent and normal at any point of a parabola meet the axis at points which are equidistant from its focus.

16. Prove that only one tangent can be drawn to a parabola in a specified direction.

17. Find the coordinates of the point of contact of the tangent to the parabola $y^2 = 4ax$ which makes an angle of θ with its axis.

18. Find the equation of the normal to the parabola $y^2 = 4ax$ which makes an angle of 60° with the axis in an anti-clockwise direction.

19. Prove that the normal to the parabola $y^2 = 4ax$ which cuts its axis at a point distant c from the vertex, is inclined to the axis at an angle $\tan^{-1}\sqrt{\left(\frac{c-2a}{a}\right)}$.

20. Find the equations of the tangents to the parabola $x^2 = 4y$ which are parallel and perpendicular to the line $y = 2x+5$.

21. The tangent at a point P on a parabola cuts the axis in T. PN is drawn perpendicular to the axis. Show that $PN^2 = 2aTN$, where $4a$ is the latus rectum.

22. The tangent and normal at a point P of a parabola whose focus is S meet the directrix in H and K. PN is drawn perpendicular to the axis. Prove that $HK.PN = 2SP^2$.

23. Find the coordinates of the point on the parabola $y^2 = 20x$ at which the tangent makes an intercept of length 125 on the x-axis.

24. If F_1, F_2 are points on the axis of a parabola equidistant from its focus, prove that the difference of the squares of the perpendiculars from F_1 and F_2 on any tangent to the parabola $= 2aF_1F_2$.

25. Find the coordinates of the points of intersection of the normals to the parabola $y^2-4ax = 0$ at the points t_1, t_2.

26. Show that the chord joining any two points t_1 and t_2 on the parabola $y^2 = 4ax$ meets the tangent at the vertex in a point whose ordinate is a times the harmonic mean of t_1 and t_2.

27. Find the condition that the chord joining two points t_1 and t_2 on the parabola $y^2 = 4ax$ should touch the parabola $y^2 = 4bx$, where $b < a$.

28. Prove that the straight line joining the focus of a parabola whose vertex is A to the point at which any chord AP meets the directrix is parallel to the tangent at P.

29. If (x_1, y_1), (x_2, y_2) are the coordinates of the extremities of a focal chord of the parabola $y^2 = 4ax$, prove that $x_1x_2 = a^2$ and $y_1y_2 = -4a^2$

30. Find the coordinates, in terms of θ, of the extremities of a focal chord of the parabola $y^2 = 4ax$ which makes an angle of θ with the axis.

31. PSQ is a focal chord of a parabola and PM, QM' are perpendicular to the directrix. Show that SM is perpendicular to SM'.

32. Prove that the normals at the extremities of a focal chord of a parabola intersect at right angles.

33. Find the locus of the intersection of the tangents to the parabola at two variable points t_1 and t_2, when (1) t_1+t_2, (2) $t_1 t_2$, (3) $t_1^2+t_2^2$, (4) $\frac{1}{t_1}+\frac{1}{t_2}$ are each in turn taken as being constant.

34. The tangents to the parabola $y^2 = 4ax$ at the points t_1 and t_2 include an angle of 45°. Show that $t_1 t_2+t_1-t_2+1 = 0$.

35. Find the locus of the point of intersection of two tangents to the parabola $y^2 = 4ax$ which include a fixed angle α.

36. Two tangents to a parabola are inclined to its axis at angles which are complementary. Prove that their chord of contact cuts the axis in a fixed point.

37. TP, TQ are tangents to the parabola $y^2 = 4ax$. TK is perpendicular to the axis and the perpendicular from T to PQ meets the axis in M. Show that $KM = 2a$.

38. Show that the locus of the points of intersection of two tangents to the parabola $y^2 = 4ax$ which, with the tangent at its vertex, form a triangle of constant area S is $x^2(y^2-4ax) = 4S^2$.

39. Prove that the area of a triangle inscribed in a parabola is twice that of the triangle formed by the tangents at its vertices.

40. The tangents at two points on a parabola meet at the point (x_1, y_1) and the normals at these points meet at the point (x_2, y_2). Show that $ay_2+x_1 y_1 = 0$.

41. Prove that the length of the chord of contact of tangents to the parabola $y^2 = 4ax$ from the point (h, k) is

$$\frac{1}{a}\sqrt{\{(k^2+4a^2)(k^2-4ah)\}}.$$

42. PQ is a normal chord of a parabola and the tangent at

the point P on it makes an angle of θ with the axis. Show that the angle between the tangents at P and Q is $\tan^{-1}(2\tan\theta)$.

43. What are the coordinates of the pole of the line

$$x\cos\alpha+y\sin\alpha = p$$

with regard to the parabola $y^2 = 4ax$?

44. Prove that the area of a triangle formed by two tangents from an external point (h, k) to the parabola $y^2 = 4ax$ and its chord of contact is $\frac{1}{2a}(k^2-4ah)^{\frac{3}{2}}$.

45. Prove that the polar of the point whose coordinates are $\left[-\frac{a}{2}(t\;+t\;+4), -\frac{4a}{t_1+t_2}\right]$ bisects perpendicularly the chord joining the points t_1 and t_2 of the parabola $y^2 = 4ax$.

46. From the vertex of a parabola any chord AP is drawn and the perpendicular through P to AP meets the axis at R. PN is drawn perpendicular to the axis. Show that $NR = 4a$.

47. AZ is the perpendicular from the vertex of a parabola $y^2 = 4ax$ to a tangent and meets the curve in Q. Show that $AZ.AQ = 4a^2$.

48. Two perpendicular chords AP, AQ through the vertex of a parabola $y^2 = 4ax$ meet the curve in P and Q. PN, QM are drawn perpendicular to the axis. Show that $PN.QM = 16a^2$.

49. Prove that the chord of a parabola $y^2 = 4ax$ which is a normal at the point $(2a, 2\sqrt{2}a)$ subtends a right angle at the vertex.

50. A chord of a parabola moves so as to subtend a right angle at its vertex. Prove that it always intersects the axis at a fixed point.

51. TP, TQ are two tangents to a parabola $y^2 = 4ax$. If the coordinates of T are (h, k), show that the equation of the lines AP, AQ is $hy^2 = 2x(ky-2ax)$, A being the vertex of the parabola.

52. AP, AQ are any two perpendicular chords through the vertex A of a parabola. Prove that the tangents to the parabola at P and Q intersect on a fixed straight line perpendicular to the axis.

53. Show that every chord of the parabola $y^2 = 4ax$ which subtends a right angle at the point $(at^2, 2at)$ on the curve, passes through the fixed point $a(t^2+4), -2at$.

54. Prove that, if the lengths of two normals from a point to a parabola are equal, that point must be on the axis.

55. A tangent to a parabola $y^2 = 4ax$ makes an angle θ with its axis. Show that its distance from the parallel normal to the parabola is $a\sec^2\theta\operatorname{cosec}\theta$.

56. Find the locus of the point of intersection of two tangents to a parabola which cut off an intercept of constant length on the tangent at its vertex.

57. The normal to a parabola cuts the axis in G. Show that a line through G perpendicular to PG always touches an equal parabola.

58. PQ is a normal chord to a parabola. The tangents to it at P and Q meet in T. Prove that P and T are equidistant from the directrix.

59. TP and TQ are two tangents to a parabola and the perpendicular from T to PQ passes through a fixed point. Find the locus of T.

60. If $t_1 = rt_2 = r^2t_3$, show that the tangents to the parabola $y^2 = 4ax$ at the points t_1 and t_3 meet on the line $x = at_2^2$.

12. Diameters of a parabola

It will first be proved that the locus of the middle points of a system of parallel chords of a parabola is a straight line.

Take any two points t_1, t_2 on the parabola $y^2 = 4ax$.

Let QQ' be the chord joining them whose equation is

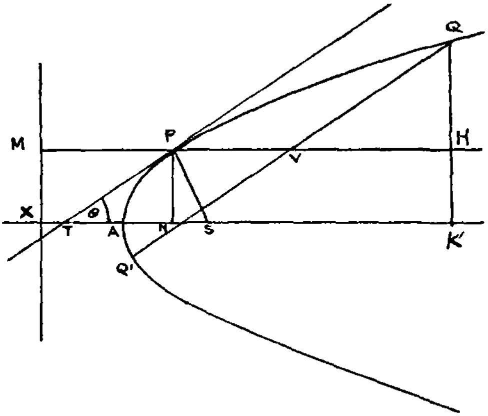
$$x-\tfrac{1}{2}(t_1+t_2)y+at_1t_2 = 0.$$

The gradient of QQ' is $\dfrac{2}{t_1+t_2}$. Provided that the chord is always in the same direction, this gradient remains the same. Hence $t_1+t_2 = c$, where c is a constant.

Let Y be the ordinate of V, the middle point of the chord.

Then $$Y = \tfrac{1}{2}(2at_1+2at_2) = a(t_1+t_2) = ac.$$

Thus the locus of V is a line parallel to the x-axis, i.e. the axis of the parabola.



The tangents at Q and Q', the points t_1, t_2, meet at a point whose ordinate is $a(t_1+t_2) = ac$.

∴ this point where they meet is on the locus of the middle points of the parallel chords.

We can now, therefore, define as a *Diameter* of a parabola the line which bisects each one of a system of parallel chords.

As already shown, a diameter

(1) *is parallel to the axis of the parabola;*

(2) *goes through the point where the tangents at the extremities of any one of the chords intersect.*

Where the diameter $y = ac$ of the parabola cuts the curve, we have $x = \dfrac{ac^2}{4}$, $y = ac$. The tangent at this point, whose parameter is $\dfrac{c}{2}$, has for its equation

$$x-\frac{c}{2}y+a\frac{c^2}{4} = 0.$$

The gradient of this line is $\frac{2}{c}$, which is the same as that of the chords of the parallel system.

Hence a diameter of a parabola

(3) *cuts the curve in a point where the tangent is parallel to the chords which the diameter bisects.*

13. Properties of diameters

If a diameter through any point P on a parabola bisects a chord QQ' in V, then $\mathbf{QV^2 = 4SP.PV}$.

With the figure of the previous article, let P be any point t on the parabola whose equation is $y^2 = 4ax$.

Let the tangent at P cut the axis of the parabola in T and make an angle θ with it.

Let the diameter PV of the system of chords parallel to QQ' cut the directrix in M, and produce MPV to cut the ordinate of Q, QK, in H.

Draw PN perpendicular to the axis and join SP.

Then the equation of the tangent at P, the point t, being

$$x - ty + at^2 = 0,$$

$$\therefore \tan\theta = \frac{1}{t} \quad \text{and} \quad t = \cot\theta.$$

$$\therefore PN = 2at = 2a\cot\theta \quad \text{and} \quad AN = at^2 = a\cot^2\theta.$$

Now $SP = PM = NX = XA + AN = a + a\cot^2\theta$

$$= \mathbf{a\,cosec^2\theta}.$$

Also $QK = QH + HK = QH + PN = QV\sin\theta + 2a\cot\theta$.

And $QK^2 = 4AS.AK = 4AS(AN + PV + VH)$

$$= 4a(a\cot^2\theta + PV + QV\cos\theta).$$

$$\therefore (QV\sin\theta + 2a\cot\theta)^2 = 4a^2\cot^2\theta + 4aPV + 4aQV\cos\theta;$$

whence $QV^2\sin^2\theta = 4aPV$.

$$\therefore \frac{QV^2}{PV} = 4a\,\mathrm{cosec}^2\theta \equiv 4SP.$$

$$\therefore \mathbf{QV^2 = 4SP.PV}.$$

14. Conormal points of a parabola

As shown in Art. 5, the equation of the normal to a parabola $y^2 = 4ax$ at the point t may be written

$$at^3+t(2a-x)-y = 0.$$

If this normal passes through a fixed point (h,k), we have

$$at^3+t(2a-h)-k = 0. \tag{1}$$

This is a cubic equation in t, and if its three roots are t_1, t_2, t_3, then the equation implies that three normals will pass through the fixed point (h,k) and that the points t_1, t_2, t_3 will be the three points on the parabola at which they are the normals.

Three points on a parabola, the normals at which meet in a point, are called *Conormal Points*.

Certain relations exist between such points. We observe, for instance, that there is no coefficient of the t^2 term in the equation (1).

$$\therefore\ t_1+t_2+t_3 = 0.$$

Hence $$2at_1+2at_2+2at_3 = 0,$$

i.e. *the algebraic sum of the ordinates of three conormal points on a parabola is zero.*

Note that if we were given that the normals at three points on a parabola were concurrent, i.e. that they were conormal points, then the coordinates of the point where they meet are given by the relations

$$\frac{2a-h}{a} = t_1t_2+t_2t_3+t_3t_1 \quad \text{and} \quad \frac{k}{a} = t_1t_2t_3.$$

The following are typical theorems connected with conormal points of a parabola.

(1) The normals drawn from the extremities of a system of parallel chords of a parabola meet on a fixed normal.

Suppose the extremities of one parallel chord of the system are the points t_1 and t_2.

The equation of the chord is $x-\frac{1}{2}(t_1+t_2)y+at_1t_2 = 0.$

Its gradient is fixed. $\therefore\ t_1+t_2 = \lambda$, say, where λ is some constant.

Let t_3 be the point conormal with t_1 and t_2.

Then $t_1+t_2+t_3 = 0$, i e. $t_3 = -(t_1+t_2) = -\lambda$.

$\therefore\ t_3$ is a fixed point, and the normals at t_1 and t_2 accordingly meet on a fixed normal at this point.

(2) If the normals at two points of a parabola meet on the curve itself, the chord joining them will always pass through a fixed point on the axis.

Suppose the two points are t_1 and t_2.

Where the chord $x-\frac{1}{2}(t_1+t_2)y+at_1t_2 = 0$ cuts $y = 0$, we have $x = -at_1t_2$.

From the equation $at^3+(2a-h)t-k = 0$ we have

$$t_1t_2t_3 = \frac{k}{a}.$$

But in this case $k = 2at_3$. $\therefore\ t_1t_2 = 2$.

$\therefore$ the chord cuts the axis at the point $(-2a, 0)$.

(3) If two of the normals passing through a fixed point are perpendicular, the point will lie on the parabola whose equation is $y^2 = a(x-3a)$.

The gradients of the normals to the parabola $y^2 = 4ax$ at the points t_1 and t_2 are $-t_1$ and $-t_2$ respectively. If these normals are perpendicular, $t_1t_2 = -1$.

If t_3 be the third conormal point, we have, as in the last example, $\frac{k}{a} = t_1t_2t_3 = -t_3$.

$$\text{And}\quad \frac{2a-h}{a} = t_1t_2+t_3(t_1+t_2) = -1-t_3^2 = -1-\frac{k^2}{a^2}.$$

$\therefore$ the locus of (h,k) is $\frac{y^2}{a^2} = -3+\frac{x}{a}$, i.e.

$$y^2 = a(x-3a).$$

N.B. The normals at the ends of a focal chord are perpendicular. This locus is therefore that of the points of intersection of normals at the extremities of focal chords.

(4) If $t_1 t_2 = -2$, find the locus of the point of intersection of the normals at t_1 and t_2.

Proceeding as in the last example, we should have

$$\frac{2a-h}{a} = -2-\frac{k^2}{a^2};$$

whence the required locus is $y^2 = a(x-4a)$, which always passes through the fixed point $(4a, 0)$ on the axis.

15. The intersection of a circle and a parabola

The general equation of a circle being

$$x^2+y^2+2gx+2fy+c = 0,$$

if we substitute in this $x = at^2$, $y = 2at$, we obtain the four points t_1, t_2, t_3, t_4 in which the circle and the parabola $y^2 = 4ax$ intersect, for

$$a^2t^4+2at^2(g+2a)+4aft+c = 0$$

is a quartic and therefore gives four points of intersection between the circle and the parabola.

Using the relations between the roots and the coefficients of the equation, we have

$$t_1+t_2+t_3+t_4 = 0 \quad \text{and} \quad 2a(t_1+t_2+t_3+t_4) = 0.$$

Hence the four points of intersection of a circle and a parabola $y^2 = 4ax$ are such that *the algebraical sum of the ordinates of the four points of intersection is zero.*

The chord joining two points t_1, t_2 is

$$x-\tfrac{1}{2}(t_1+t_2)y+at_1 t_2 = 0.$$

And that joining t_3 and t_4 is

$$x-\tfrac{1}{2}(t_3+t_4)y+at_3 t_4 = 0.$$

But $t_1+t_2 = -(t_3+t_4)$.

$\therefore$ the gradients of the chords are the same except in sign. This implies that *the chords are equally inclined to the axis.*

Suppose that three of the points of intersection of the circle and the parabola, say t_1, t_2, and t_3, were conormal points. Then, as shown in the previous article,

$$t_1+t_2+t_3 = 0.$$

But if t_4 be the fourth point of intersection,

$$t_1+t_2+t_3+t_4 = 0.$$

$\therefore$ in this case $t_4 = 0$, i e. t_4 is the vertex.

Hence, if a circle passes through three conormal points of a parabola, it must also pass through the vertex.

16. Equation of two tangents to a parabola

Let (h, k) be a fixed point not on the parabola $y^2 = 4ax$.

The method of finding the equation of the tangents from (h, k) to the parabola is the same as that employed in the case of the circle.

Let (X, Y) be any point on a tangent from (h, k) to the parabola $y^2 = 4ax$

The coordinates of a point dividing the line joining (h, k) and (X, Y) in the ratio $p \cdot q$ are

$$\frac{pX+qh}{p+q} \quad\text{and}\quad \frac{pY+qk}{p+q}.$$

Substitute these coordinates in the equation $y^2 = 4ax$.

Then $$\left(\frac{pY+qk}{p+q}\right)^2 = 4a\left(\frac{pX+qh}{p+q}\right);$$

whence

$$\frac{p^2}{q^2}(Y^2-4aX)+\frac{2p}{q}(Yk-2aX-2ah)+k^2-4ah = 0.$$

Now if the line joining (h, k) and (X, Y) is a tangent to the parabola, the roots of this equation must be equal.

$$\therefore\ (Yk-2aX-2ah)^2 = (Y^2-4aX)(k^2-4ah),$$

which is the equation of the two tangents from the point (h, k) to the parabola $y^2 = 4ax$.

As an aid to memory we notice that the left-hand side is the terms of the equation of the tangent at (h, k) and the right-hand side the product of the terms of the equation of the parabola and the terms of the same equation with (h, k) substituted in it.

If the tangents were at right angles, then the coefficients of X^2 and Y^2 in the equation just obtained would have their sum zero, i.e.

$$k^2+4a^2-k^2+4ah = 0;$$

whence

$$h = -a;$$

i.e. the point must lie on the directrix, and we have a proof of the theorem that if two tangents to a parabola are perpendicular, their point of intersection lies on the directrix.

17. Envelopes

When the coefficients of a line whose equation is $lx+my = 1$ are connected by a relation which enables us to express this equation in terms of a single variable parameter, the condition *that two roots of this equation should be equal will give the envelope of the line, i.e. a curve which it will always touch.*

Consider the equation $x-ty+at^2 = 0$.

This is an equation of the second degree in t, and the condition that its roots should be equal is

$$y^2 = 4ax,$$

as we should expect.

Again, suppose that we were given the line $lx+my = 1$ and $am^2 = l$, then $am^2x+my-1 = 0$ and the condition for equal roots is $y^2+4ax = 0$, which means that the parabola $y^2+4ax = 0$ is always enveloped, or touched, by the line $lx+my = 1$ so long as $am^2 = l$.

A few typical examples will further illustrate the process involved.

(1) P is a point on the parabola $y^2 = 4ax$ and PL, PM are drawn perpendicular to the tangent at the vertex and the axis. Find the envelope of LM.

The equation of LM is clearly $\frac{x}{at^2}+\frac{y}{2at} = 1$ if P is the point t.

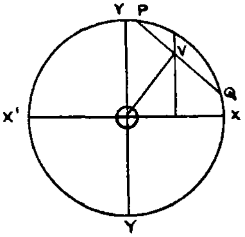
This equation may be rewritten $2at^2-ty-2x = 0$. And the condition for equal roots gives

$$y^2+16ax = 0,$$

which is the equation of the envelope of LM.

(2) The middle point of a chord of a circle is always on a fixed straight line; show that it envelops a parabola.

Take a diameter XOX' perpendicular to the fixed line as x-axis and YOY' parallel to it as y-axis. Let the equation of the fixed line be $x = k$.



Let the equation of the chord PQ be $lx+my = 1$.

At V, where it cuts $x = k$, we have $y = \frac{1-lk}{m}$.

The line joining the centre to this point, whose equation is

$$\frac{x}{k} = \frac{my}{1-lk},$$

is perpendicular to $lx+my = 1$.

$$\therefore \frac{1-lk}{mk}\cdot\frac{l}{m} = 1.$$

$$\therefore m^2 = \frac{l(1-lk)}{k} = \frac{l}{k}-l^2.$$

$$\therefore (1-lx)^2 = m^2y^2 = \left(\frac{l}{k}-l^2\right)y^2.$$

$$\therefore l^2(x^2+y^2)-l\left(2x+\frac{y^2}{k}\right)+1 = 0.$$

The condition for equal roots gives

$$\left(2x+\frac{y^2}{k}\right)^2 = 4(x^2+y^2),$$

which reduces to $$\frac{y^2}{k^2}+\frac{4x}{k} = 4,$$

i.e. $$y^2 = -4k(x-k),$$

which is the required envelope and it is a parabola whose latus rectum is $4k$ and which passes through the point $(k, 0)$ which is its vertex.

(3) The normals at three points of a parabola meet at a point on a fixed diameter. Show that the triangle formed by the points envelops a parabola.

With the usual notation let $y = k$ be the fixed diameter on which the three normals meet, then $\frac{k}{a} = t_1 t_2 t_3$ (Art. 14).

The equation of the line joining t_2, t_3, say, is

$$x-\frac{t_2+t_3}{2}y+at_2 t_3 = 0.$$

But as $t_1+t_2+t_3 = 0$ and $t_1 t_2 t_3 = \frac{k}{a}$, this may be written

$$x+\tfrac{1}{2}t_1 y+\frac{k}{t_1} = 0,$$

i.e. $$t_1^2 y+2xt_1+2k = 0,$$

from which the condition for equal roots gives $x^2 = 2ky$.

$\therefore$ the chord joining the two points t_2 and t_3 envelops the parabola $x^2 = 2ky$.

The symmetry of the work shows that we should get the same envelope had we taken either the chord joining the points t_1 and t_2 or that joining the points t_3 and t_4.

Hence the theorem is established.

(4) *PNP'* is a double ordinate of a parabola. Find the envelope of all chords of the parabola which are bisected by it.

Let the equation of the parabola be $y^2 = 4ax$ and that of the double ordinate $x = c$.

Then, if $x-\frac{1}{2}(t_1+t_2)y+at_1 t_2 = 0$ be the equation of the chord of the parabola which is bisected by the ordinate $x = c$, $c = \frac{1}{2}(at_1^2+at_2^2)$.

Let $p = t_1+t_2$ and $q = t_1 t_2$.

Then
$$x-\tfrac{1}{2}py+aq = 0$$
and
$$2c = a(t_1^2+t_2^2) = a(p^2-2q).$$
$$\therefore\ x-\frac{p}{2}y+\frac{ap^2}{2}-c = 0,$$
of which the envelope is
$$\frac{y^2}{4} = 4\frac{a}{2}(x-c),$$
i.e.
$$y^2 = 8a(x-c).$$

(5) If P lies on the parabola $y^2+ax = 0$, find the envelope of its polar with regard to the circle $x^2+y^2 = 4ax$.

Let P be the point $-\frac{a}{4}t^2,\ \frac{a}{2}t$.

The equation of the polar of P with regard to the circle is
$$-x\frac{a}{4}t^2+\frac{ya}{2}t-2a\left(x-\frac{a}{4}t^2\right) = 0,$$
i.e.
$$t^2(2a-x)+2yt-8x = 0,$$
from which we get the equation of the envelope to be
$$y^2+8x(2a-x) = 0.$$

(6) Circles are drawn on chords of the parabola $y^2 = 4ax$ as diameters, the chords being parallel to $y = mx$. Prove that the envelope of these circles is
$$(my-2a)^2 = 4a(1+m^2)(x+a). \qquad \text{[B.]}$$

The equation of a chord of the parabola is
$$x-\tfrac{1}{2}(t_1+t_2)y+at_1 t_2 = 0.$$

Then
$$t_1+t_2 = \frac{2}{m}.$$

Let
$$t_1 t_2 = \lambda.$$

The equation of a circle on the line joining t_1 and t_2 as diameter is

$$(x-at_1^2)(x-at_2^2)+(y-2at_1)(y-2at_2) = 0,$$

i.e. $$x^2+y^2-ax(t_1^2+t_2^2)-2ay(t_1+t_2)+a^2t_1^2t_2^2+4a^2t_1t_2 = 0,$$

i.e. $$(x^2+y^2)m^2-ax(4-2\lambda m^2)-4amy+a^2m^2\lambda^2+4a^2\lambda m^2 = 0,$$

i.e. $$\lambda^2a^2m^2+2am^2\lambda(x+2a)+m^2(x^2+y^2)-4ax-4amy = 0,$$

whose envelope is

$$m^2(x+2a)^2 = m^2(x^2+y^2)-4ax-4amy,$$

which reduces to

$$(my-2a)^2 = 4a(1+m^2)(x+a).$$

18. Typical examples

Ex. 1. From any point on the line $x+y = 0$ tangents are drawn to the parabola $y^2 = 4ax$; prove that the locus of the middle point of their chord of contact is the parabola

$$y^2 = 2a(x-y). \qquad \text{[D.]}$$

Let $(-h,h)$ be any point on the line $x+y = 0$; its polar with regard to the parabola $y^2 = 4ax$ is

$$yh = 2a(x-h). \qquad (1)$$

Since the line joining the point $(-h,h)$ to the middle point (X,Y) of the chord of contact is parallel to the axis, $\therefore\ Y = h$.

Since (X,Y) satisfies equation (1),

$$\therefore\ Yh = 2a(X-h).$$

Putting $h = Y$, this becomes

$$Y^2 = 2a(X-Y),$$

which is the required locus.

Ex. 2. If two of the normals from the point $P(x_1,y_1)$ to the parabola $y^2-4ax = 0$ are perpendicular to each other, prove that the foot of the third normal through P has coordinates $(x_1-3a, -2y_1)$. [C.]

The conormal points t_1, t_2, t_3 are given by the equation

$$at^3+t(2a-x_1)-y_1 = 0 \quad \text{and} \quad t_1t_2 = -1.$$

$$\therefore \frac{2a-x_1}{a} = t_1t_2+t_2t_3+t_3t_1 = -1-t_3^2.$$

$$\therefore at_3^2 = x_1-3a.$$

Again, $$\frac{y_1}{a} = t_1t_2t_3 = -t_3.$$

$$\therefore 2at_3 = -2y_1.$$

Ex. 3. P_1, P_2, P_3 are the vertices of any one of a series of triangles inscribed within a given parabola, such that their centroids lie on the curve's axis. Prove that the middle points of the sides of the triangle formed by the tangents to the parabola at P_1, P_2, P_3 lie on a fixed parabola whose vertex and axis (produced backwards) coincide with those of the given parabola. [O.]

Let P_1, P_2, P_3 be the points t_1, t_2, and t_3.

The ordinate of the centroid of the triangle $P_1P_2P_3$ is

$$\frac{2a(t_1+t_2+t_3)}{3},$$

and as this lies on the axis, $t_1+t_2+t_3 = 0$.

The tangents at P_1 and P_2 intersect in the point $at_1t_2, a(t_1+t_2)$. The tangents at P_1 and P_3 similarly intersect in the point $at_1t_3, a(t_1+t_3)$.

The coordinates of the middle point of the line joining these two are $\frac{1}{2}a(t_1t_2+t_1t_3)$, $\frac{1}{2}a(2t_1+t_2+t_3)$. But as $t_2+t_3 = -t_1$, these may be written
$$-\tfrac{1}{2}at_1^2, at_1.$$

So the other pair of points are $(-\frac{1}{2}at_2^2, at_2)$, $(-\frac{1}{2}at_3^2, at_3)$. And all three clearly lie on the parabola $y^2+2ax = 0$ which has the same vertex and axis (produced backwards) as the parabola $y^2 = 4ax$.

Ex. 4. Prove that the normals at the points where the straight line $px+qy = 1$ meets the parabola $y^2 = 4ax$ intersect on the normal at the point $\left(\frac{4aq^2}{p^2}, \frac{4aq}{p}\right)$. [D.]

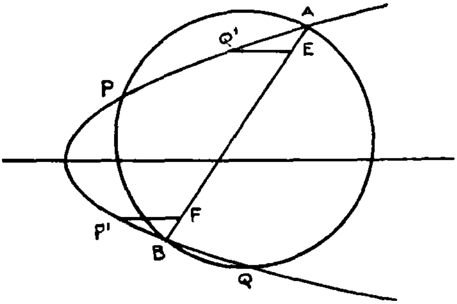
If the two points are t_1 and t_2, then t_1 and t_2 are the roots of the equation $pat^2+2qat-1 = 0$.

$$\therefore\ t_1+t_2 = -\frac{2q}{p},\qquad t_1 t_2 = -\frac{1}{pa}.$$

If t_3 be the foot of the third normal, $t_1+t_2+t_3 = 0$.

$\therefore\ t_3 = \frac{2q}{p}$, and the point is $(at_3^2, 2at_3)$, i.e. $\left(\frac{4aq^2}{p^2}, \frac{4aq}{p}\right)$.

Ex. 5. A circle is drawn on a chord AB of the parabola $y^2 = 4ax$ as diameter. It meets the parabola again at P, Q;



P', Q' are the reflections of P, Q in the axis. $P'F$ and $Q'E$ are drawn parallel to the axis and meet AB at F and E. Prove that $P'F = Q'E = 4a$. [O. and C.]

Let A, B, P, Q be the points t_1, t_2, t_3, t_4.

Then P' is the point $-t_3$ and Q' is the point $-t_4$.

The equation of the circle on AB as diameter is

$$(x-at_1^2)(x-at_2^2)+(y-2at_1)(y-2at_2) = 0.$$

In this put $x = aT^2$, $y = 2aT$; then t_1, t_2, t_3, t_4 are the roots of $T^4+T^2(4-t_1^2-t_2^2)-4T(t_1+t_2)+4t_1 t_2+t_1^2 t_2^2 = 0$.

$$\therefore\ t_1(t_2+t_3+t_4)+t_2(t_3+t_4)+t_3 t_4 = 4-t_1^2-t_2^2$$

and

$$t_1+t_2+t_3+t_4 = 0.$$

$$\therefore\ -t_1^2-t_2(t_1+t_2)+t_3 t_4 = 4-t_1^2-t_2^2.$$

$$\therefore\ t_3 t_4-t_1 t_2 = 4.$$

Where $P'F$ cuts AB, i.e. where $y = -2at_3$ cuts

$$x-\tfrac{1}{2}(t_1+t_2)y+at_1t_2 = 0,$$

we have

$$x = -at_3(t_1+t_2)-at_1t_2.$$

$$\therefore\ P'F = at_3^2+at_3(t_1+t_2)+at_1t_2$$

$$= at_3^2-at_3(t_3+t_4)+at_1t_2$$

$$= a(t_1t_2-t_3t_4) = 4a.$$

Similarly,

$$Q'E = at_4^2+at_4(t_1+t_2)+at_1t_2$$

$$= a(t_1t_2-t_3t_4) = 4a.$$

$$\therefore\ P'F = Q'E = 4a.$$

Ex. 6. Find the locus of the point of intersection of two tangents to a parabola which cut off an intercept of constant length on its axis.

The equation of the tangents from a point (h,k) to the parabola $y^2 = 4ax$ is

$$[yk-2a(x+h)]^2 = (y^2-4ax)(k^2-4ah).$$

Where this cuts $y = 0$ we have

$$4a^2(x+h)^2+4ax(k^2-4ah) = 0,$$

i.e.

$$ax^2+x(k^2-2ah)+ah^2 = 0$$

If x_1, x_2 be the roots of this equation,

$$x_1+x_2 = \frac{2ah-k^2}{a} \quad \text{and} \quad x_1x_2 = h^2.$$

$$\therefore\ (x_1-x_2)^2 = \frac{(2ah-k^2)^2}{a^2}-4h^2.$$

Hence, if the intercept x_1-x_2 on the axis is equal to c, a constant,

$$c^2a^2 = (2ah-k^2)^2-4h^2a^2,$$

and the required locus is

$$c^2a^2 = y^4-4axy^2 = y^2(y^2-4ax).$$

Ex. 7. The normals at three conormal points on a parabola meet the axis in points G_1, G_2, G_3, and AG_2 is the arithmetic mean of AG_1 and AG_3, where A is the vertex of the parabola. Find the locus of the point where the three normals meet.

Let the normals at the points t_1, t_2, t_3 meet in (h, k).

Then t_1, t_2, and t_3 are the roots of the equation

$$at^3+t(2a-h)-k = 0.$$

By data, $$a(t_1^2+t_3^2) = 2at_2^2.$$

Also $$t_1+t_3 = -t_2.$$

$$\therefore\ 2t_2^2 = t_1^2+t_3^2 = (t_1+t_3)^2-2t_1t_3 = t_2^2-2t_1t_3.$$

$$\therefore\ t_2^2 = -2t_1t_3.$$

Now $$\frac{2a-h}{a} = t_1t_3+t_2(t_1+t_3) = t_1t_3-t_2^2 = -\tfrac{3}{2}t_2^2$$

and $$\frac{k}{a} = t_1t_2t_3 = -\frac{t_2^3}{2}.$$

$$\therefore \frac{4k^2}{a^2} = t_2^6 = \frac{8}{27}\left(\frac{h}{a}-2\right)^3;$$

whence the required locus is

$$27ay^2+2(2a-x)^3 = 0.$$

Ex. 8. Two parabolas have a common vertex A and axis and an ordinate NPQ meets them in P and Q. The tangent at P meets the outer parabola in U and V and AU, AV meet the ordinate NPQ in H and K. Prove that

$$(1)\ NQ^2 = NH.NK, \quad (2)\ \frac{2}{NP} = \frac{1}{NH}+\frac{1}{NK}.$$

Let the two parabolas be $y^2 = 4ax$ and $y^2 = 4\alpha x$ and P be a point t on the former.

The tangent at P is $x-ty+at^2 = 0$.

The lines joining the vertex $(0, 0)$ to the intersection of this line with $y^2 = 4\alpha x$ have as their equation

$$y^2 = 4\alpha x\left(\frac{y}{at}-\frac{x}{at^2}\right).$$

Where these lines cut $x = at^2$, we have

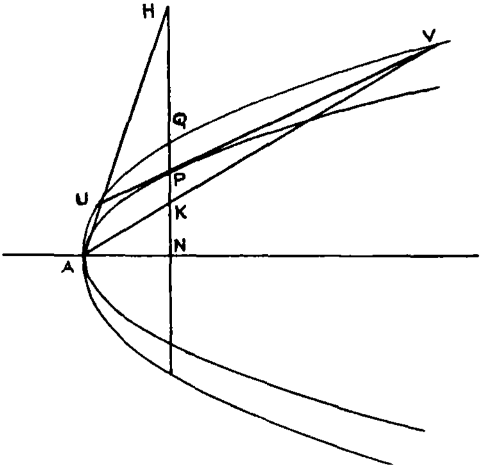
$$y^2 = 4\alpha at^2\left(\frac{y}{at}-1\right).$$

NH, NK are the roots of this equation.

$$\therefore\ NH.NK = 4\alpha at^2.$$

But where $x = at^2$ cuts $y^2 = 4\alpha x$ we have $NQ^2 = 4\alpha at^2$.

$$\therefore\ NH.NK = NQ^2.$$



Again,

$$\frac{1}{NH}+\frac{1}{NK} = \frac{NH+NK}{NH.NK} = \frac{4\alpha t}{4\alpha at^2} = \frac{1}{at} = \frac{2}{NP}.$$

Ex. 9. A right-angled triangle whose hypotenuse is always in a fixed direction is inscribed in a parabola. Prove that the locus of its centroid is another parabola.

Let the parabola be $y^2 = 4ax$, t_1 and t_2 the extremities of the hypotenuse of the inscribed triangle, and t its third vertex.

The gradient of $x-\frac{1}{2}(t_1+t_2)y+at_1t_2 = 0$ is fixed.

$$\therefore\ t_1+t_2 = \text{a constant} = c\text{, say.}$$

Let (X, Y) be the coordinates of the centroid. Then

$$\frac{3X}{a} = t_1^2+t_2^2+t^2 \quad\text{and}\quad \frac{3Y}{2a} = t_1+t_2+t = c+t.$$

But since the lines joining t and t_1, t and t_2, are perpendicular,

$$\therefore\ \frac{2}{t+t_1}\cdot\frac{2}{t+t_2} = -1.$$

$$\therefore\ t^2+t(t_1+t_2)+t_1t_2+4 = 0.$$

$$\therefore\ t_1t_2 = -(t^2+ct+4).$$

Now
$$\frac{3X}{a} = (t_1+t_2)^2-2t_1t_2+t^2$$
$$= c^2+8+2ct+3t^2$$
$$= c^2+8+2c\left(\frac{3Y}{2a}-c\right)+3\left(\frac{3Y}{2a}-c\right)^2,$$

which is clearly of the form $y^2+px+qy+r = 0$ and therefore represents a parabola.

Ex. 10. Find the locus of the centres of circles which touch a given line and a given circle.

Let the centre of any circle which touches the circle $x^2+y^2 = a^2$ and the line $x = k$ be the point X, Y. Then the radius of the circle is $k-X$.

$$\therefore\ a+k-X = \sqrt{(X^2+Y^2)},$$

as the circle touches $x^2+y^2 = a^2$.

$$\therefore \text{ the locus is }\ (a+k)^2-2(a+k)X = Y^2,$$

i.e.
$$Y^2 = -2(a+k)\left[X-\frac{a+k}{2}\right],$$

which is clearly a parabola of latus rectum $2(a+k)$ whose vertex is the point $\left(\frac{a+k}{2}, 0\right)$.

EXAMPLES 11

1. The sides of a quadrilateral inscribed in a parabola make angles of $\alpha, \beta, \gamma, \delta$ with its axis. Show that

$$\cot\alpha+\cot\gamma = \cot\beta+\cot\delta.$$

2. Find the locus of the middle points of chords of a parabola which pass through the focus.

3. Find the locus of the middle points of chords of a parabola which are of given length c.

4. Find the locus of the middle points of normal chords of a parabola.

5. Find the locus of the middle points of chords of a parabola which pass through a fixed point (h, k).

6. Prove that two parabolas having the same focus and their axes in opposite directions intersect at right angles.

7. A line is drawn through the fixed point (h, k) at right angles to the tangent at a point P on the parabola $y^2 = 4ax$. It meets the diameter through P in Q. Show that the locus of Q is

$$y(x+2a-h) = 2ak.$$

8. If the diameter bisecting a focal chord of a parabola cuts the parabola in P, prove that the length of the chord is $4SP$, S being the focus.

9. The line $lx+my+n = 0$ meets the parabola $y^2 = 4ax$ at P and Q, and the lines joining $(c, 0)$ to P and Q meet the curve again at H and K. Show that the equation of HK is $nx-mcy+lc^2 = 0$. Prove also that, if PQ passes through a fixed point, HK will also pass through another fixed point. [O. and C.]

10. Find an expression for the rectangle contained by the segments of a chord of the parabola $y^2 = 4ax$ made by the point (h, k), if the chord makes an angle of α with the axis.

11. The tangents at two points P and Q of a parabola meet at T. The diameter through T cuts the parabola in U and the chord in V. Show that $TU = UV$.

12. The diameter through a point R on a parabola cuts a chord PQ in V. PH, QK are drawn parallel to the tangent at R to cut the diameter in H and K. Prove that $RH.RK = RV^2$.

13. A chord PP' of a parabola $y^2 = 4ax$ makes a constant angle α with its axis. A point Q is taken on PP' such that $PQ.QP' = k^2$, where k is a constant. Prove that the locus of Q is a parabola.

14. Obtain the equations of the tangents from the point $(2, -6)$ to the parabolas $y^2 = 8x$ and $(y-2)^2 = 8(x-3)$.

15. Find the locus of the pole of a straight line with regard to the parabola $y^2 = 4ax$ which is parallel to $lx+my = 1$.

16. From a fixed point O on a parabola $y^2 = 4ax$ chords OH, OK are drawn at right angles to each other. Find the locus of the pole of HK.

17. A chord of the parabola $y^2 = 4a(x+c)$ subtends a right angle at the origin. Show that the locus of its pole is

$$ax^2-cy^2+4a(x+c)(a+c) = 0.$$

18. Prove that the polar of any point on the circle $x^2+y^2 = a^2$ with regard to the circle $x^2+y^2 = 2ax$ always touches the parabola $y^2 = 4ax$.

19. Show that the locus of the poles with regard to the parabola $y^2 = 4ax$ of tangents to the parabola $x^2 = 4ay$ is the curve $xy+2a^2 = 0$.

20. B is a point on the axis of a parabola whose focus is S through which any chord is drawn. Show that the locus of the foot of the perpendicular from the pole of any such chord to the chord is a circle whose centre is S and radius SB.

21. Show that the polar of a fixed point with regard to a family of parabolas having a common focus and axis envelops another parabola.

22. PSP' is a focal chord of a parabola and PQ the normal chord through P. The tangents at P and Q meet in T. Prove that TP' is parallel to the axis of the parabola.

23. Find the locus of the middle points of all tangents drawn from points on the directrix of a parabola.

24. If the diameters at two points P and Q on a parabola meet the lines PR, QR joining P and Q to a third point R on the parabola, in D and E respectively, prove that the pole of PQ is the middle point of DE.

25. If the normals at two points P and Q on a parabola meet on the curve, show that the pole of PQ with regard to the parabola lies on a fixed straight line.

26. TP, TQ are tangents to the parabola $y^2 = 4ax$ and the

normals at P and Q meet on the curve at R. Find the locus of the centre of the circle circumscribing the triangle TPQ.

27. PQ is a normal chord of a parabola; prove that the tangent at P is divided by the directrix and the diameter through Q in the ratio 1 : 3.

28. Find the locus of a point from which two normals can be drawn to a parabola making complementary angles with the axis.

29. P is a point on a parabola whose focus is S. On SP as diameter a circle is described cutting the normal chord PQ in H. Find the length of PH.

30. Prove that the common tangents of the parabola $y^2 = 4ax$ and the circle $x^2+y^2+2ax = 0$ intersect in the point $(-3a, 0)$.

31. The middle point of a chord of a circle is on a fixed straight line. Show that the chord always touches a fixed parabola.

32. Show that the normals at two points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ meet at a point whose coordinates are $a(t_1^2+t_2^2+t_1 t_2)+2a$ and $-at_1 t_2(t_1+t_2)$.

33. Show that the normal at a point $(at^2, 2at)$ on the parabola $y^2 = 4ax$ meets the curve again at a point whose parameter is $-t-\dfrac{2}{t}$.

34. Show that if the normals at two points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ meet on the curve, then $t_1 t_2 = 2$.

35. The chord PQ of the parabola $y^2 = 4ax$ passes through the foot of the directrix, the point $(-a, 0)$. The tangents at P and Q meet at T and the normals at G. Show that TG is bisected by the axis.

36. A chord PQ of a parabola $y^2 = 4ax$ is drawn through the point $(-2a, 0)$; show that the normals at P and Q meet at a point on the parabola.

37. Give a reason why at least one real normal can be drawn to a parabola from any point.

38. The point R on the normal chord PQ of a parabola $y^2 = 4ax$ divides it in the ratio 1 : 3. Show that the locus of R is the parabola $y^2 = ax-3a^2$.

39. PQ is a normal chord of a parabola and the tangents at P and Q meet at T. pq is the tangent parallel to PQ. If S be the focus, show that Sq is perpendicular to ST.

40. Normals are drawn at the extremities of any chord of a parabola passing through a fixed point on its axis. Find the locus of their point of intersection.

41. From the vertex of a parabola $y^2 = 4ax$ a chord is drawn making an angle of 30° with the axis and cutting the parabola in H. The normals at two points U and V on the parabola meet at H. Prove that $UV = 8a$.

42. Find the locus of the point of intersection of the normals at two points on a parabola when the chord joining them always subtends a right angle at the vertex.

43. P is a point $(at^2, 2at)$ on the parabola $y^2 = 4ax$, and the normal at P cuts the curve again in Q. QH, QK are the other two normals from Q to the parabola. If H and K be the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ respectively, show that $t_1+t_2 = t+\frac{2}{t}$.

44. Show that the point whose coordinates are $\frac{4am^2}{l^2}$, $\frac{4am}{l}$ is conormal with the two points in which the line $lx+my = 1$ cuts the parabola $y^2 = 4ax$.

45. Prove that the feet of all normals to the parabola $y^2 = 4ax$ which pass through the point $(4a, a)$ lie on the curve

$$xy = 2a(y+a).$$

46. Find the locus of the point where three normals to a parabola meet when one of them is the bisector of the angle between the other two.

47. If P, Q, R are three conormal points on a parabola $y^2 = 4ax$ whose focus is S, and O the point of concurrence of the normals, prove that $SP.SQ.SR = a.SO^2$.

48. The three normals from a point O to a parabola make angles of θ_1, θ_2, θ_3 respectively with its axis. Show that

$$SO\cos\theta_1\cos\theta_2\cos\theta_3 = a,$$

where S is the focus of the parabola and $4a$ its latus rectum.

49. If P, Q, R are three conormal points of a parabola, show that the centroid of the triangle PQR lies on its axis.

50. The three normals at points P, Q, R on a parabola meet at O; show that the difference between the sum of their intercepts on the axis and the semi latus rectum is equal to twice the distance of O from the tangent at the vertex.

51. The normals at three conormal points on the parabola $y^2 = 4ax$ meet at a point O. If the polar of O with regard to the circle passing through the three points always passes through a fixed point, find the locus of O.

52. Find the equations of the circles which touch the parabola $y^2 = 4ax$ at two points and pass through its focus.

53. Show that a circle described on a focal chord of a parabola as diameter touches its directrix.

54. Show that a circle described on a focal radius of a parabola as diameter touches the tangent at the vertex.

55. Two circles whose centres are on the axis of a parabola touch each other and also touch the parabola. Prove that the difference of their radii is equal to the latus rectum.

56. If the axes of two parabolas are perpendicular, show that their four points of intersection lie on a circle.

57. A circle is drawn through the points where the line $lx+my+n = 0$ cuts the parabola $y^2 = 4ax$, and it touches the parabola at the point P. Find the coordinates of P and also those of the centre of the circle.

58. If PQ is a chord of the parabola $y^2 = 4ax$ which touches the parabola $y^2 = 4a(x-4a)$, show that the circle on PQ as diameter touches the first parabola.

59. If a circle touches the latus rectum of a parabola and has a chord of the parabola for its diameter, prove that this chord must touch the circle whose equation is $x^2+y^2-6ax+a^2 = 0$.

60. Circles are described on any two focal chords of a parabola PSP', QSQ' as diameters. Prove that their common chord always passes through the vertex of the parabola.

61. TP, TQ are two tangents to a parabola and any third tangent is drawn. Show that the product of the perpendiculars

from P and Q on this tangent is equal to the square of the perpendicular from T on it.

62. P, Q, R are three conormal points on a parabola and (h, k) their point of concurrence. Prove that the area of the triangle PQR is $a^{\frac{1}{2}}[4(2a-h)^3+27ak^2]^{\frac{1}{2}}$.

Formulae of Chapter V for reference

1. Equation of the chord: $\mathbf{x-\frac{t_1+t_2}{2}y+at_1t_2=0.}$
2. Equations of the tangent: $\mathbf{x-ty+at^2=0.}$
   $\mathbf{y=mx+\frac{a}{m}.}$
   $\mathbf{yy'=2a(x+x').}$
3. Equation of the normal: $\mathbf{tx+y=at^3+2at.}$
4. The tangents at t_1 and t_2 meet at the point: $\mathbf{at_1t_2, a(t_1+t_2).}$
5. Equation of polar of (h, k): $\mathbf{yk=2a(x+h).}$
6. Polar equation: $\mathbf{\frac{l}{r}=1-\cos\theta.}$
7. Polar equation of tangent at α: $\mathbf{\frac{l}{r}=\cos(\theta-\alpha)-\cos\theta.}$
8. Ordinate QV to a diameter: $\mathbf{QV^2=4SP.PV.}$
9. For three conormal points: $\mathbf{t_1+t_2+t_3=0.}$
10. For four concyclic points: $\mathbf{t_1+t_2+t_3+t_4=0.}$
11. Equation of tangents from (h, k):
$$\mathbf{(yk-2ax-2ah)^2=(y^2-4ax)(h^2-4ak).}$$

VI

THE ELLIPSE

1. Definition and equation of the ellipse

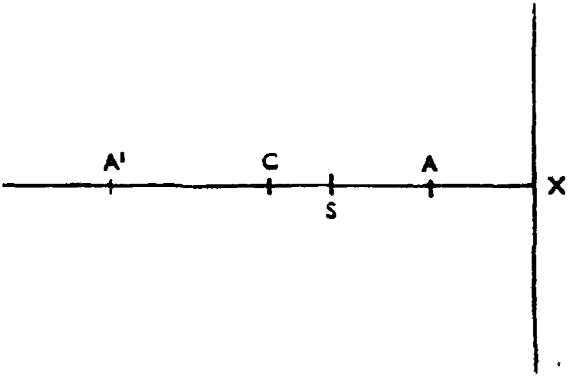
IF a plane is drawn cutting all the generators of one branch of a cone, and not at right angles to the axis of the cone, its intersection with the conical surface is a closed curve called an *Ellipse*.

An ellipse is also defined as the locus of a point which moves so that its distance from a fixed point called the *Focus* is always a constant ratio, less than unity, of its distance from a fixed straight line called the *Directrix*.

The constant ratio referred to in this definition is called the *Eccentricity* of the ellipse, which is usually denoted by the letter e. In the case of the ellipse $e < 1$.

Let S be the focus of an ellipse whose eccentricity is e.

Draw SX perpendicular to the directrix and produce it indefinitely.



Divide SX internally and externally at two points A and A' in the ratio $e:1$. Then A and A' are clearly points on the locus because they satisfy the required condition.

Bisect AA' at C and let $CA = CA' = a$.

We shall now prove that **CS = ae** and **CX =** $\mathbf{\frac{a}{e}}$.

For (1) $SA'-SA = e(A'X-AX) = eAA' = 2ae.$

But $SA'-SA = CS+CA'-CA+CS = 2CS.$

$$\therefore\ CS = ae.$$

(2) $SA'+SA = e(A'X+AX) = e(CX+CA'+CX-CA)$
$= 2eCX.$

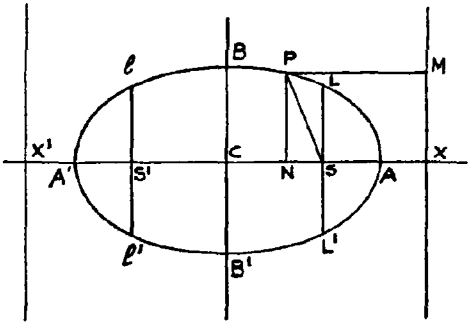
But $SA'+SA = AA' = 2a.$

$$\therefore\ CX = \frac{a}{e}.$$

Now take the point C as origin, the line $XASCA'$ as x-axis, and a perpendicular line through C to it as y-axis.

Let P be any point on the locus whose coordinates are (x, y).

Draw PN perpendicular to the x-axis and PM perpendicular to the directrix. Join SP.



From the definition of the ellipse $SP = ePM$.

$\therefore\ SP^2 = e^2PM^2.\qquad \therefore\ SN^2+PN^2 = e^2PM^2 = e^2NX^2.$

But $SN = CS-CN = ae-x$ and $NX = CX-CN = \frac{a}{e}-x.$

$$\therefore\ (ae-x)^2+y^2 = e^2\left(\frac{a}{e}-x\right)^2 = a^2-2aex+e^2x^2.$$

$$\therefore\ x^2(1-e^2)+y^2 = a^2(1-e^2).$$

$$\therefore\ \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)} = 1.$$

The points B, B' in the diagram, where this locus cuts the y-axis, are obtained by putting $x = 0$ in the equation. Hence $CB^2 = y^2 = a^2(1-e^2) = b^2$ suppose. The equation of the ellipse then becomes

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2} = \mathbf{1},$$

and $$\mathbf{b}^2 = \mathbf{a}^2(\mathbf{1}-\mathbf{e}^2).$$

ACA', BCB' are respectively the major and minor *Axes* of the ellipse of lengths $2a$ and $2b$ respectively.

From an inspection of the form of the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ certain facts can at once be deduced about the ellipse it represents.

(1) As $-x$, $-y$ can be substituted in the equation for x and y without altering its form, we infer that the curve is symmetrical about the origin and that this point is its centre.

(2) As $-x$ can be put for x, leaving y unchanged, or $-y$ for y, leaving x unchanged, without altering the form of the equation, we infer that the curve is symmetrical about the axis of y or about the axis of x.

(3) Since $y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$, there will be no real value of y for any value of x not between a and $-a$. Similarly, writing the equation $x^2 = a^2\left(1-\frac{y^2}{b^2}\right)$, there will be no real value of x for any value of y not between b and $-b$.

The curve therefore lies entirely within the rectangle formed by the lines $x = \pm a$, $y = \pm b$.

(4) The symmetry of the curve with regard to the y-axis shows that it will have two foci and two directrices.

If two chords LSL', $lS'l$ are drawn through the foci perpendicular to the major axis, each of these is a *Latus Rectum* of the ellipse.

From the diagram and the equation of the ellipse we have

$$\frac{CS^2}{a^2}+\frac{LS^2}{b^2}=1.$$

$$\therefore\ LS^2=b^2\left(1-\frac{CS^2}{a^2}\right)=b^2\left(1-\frac{a^2e^2}{a^2}\right)=b^2(1-e^2).$$

And since $b^2=a^2(1-e^2)$,

$$\therefore\ LS^2=\frac{b^4}{a^2}\quad\text{and}\quad LS=\frac{b^2}{a}.$$

The latus rectum of the ellipse is therefore of length $\dfrac{2b^2}{a}$.

From symmetry $lS'l'$ will be also of the same length.

2. Parametric coordinates of an ellipse

In the equation of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, if we put $x=a\cos\phi$, $y=b\sin\phi$, the value of the left-hand side is $\cos^2\phi+\sin^2\phi=1$, no matter what angle ϕ may be.

Hence, considering ϕ as the parameter, the coordinates of any point on an ellipse can be expressed as $a\cos\phi$, $b\sin\phi$. Thus, when we speak of 'the point ϕ on an ellipse', we mean that point whose coordinates are $a\cos\phi$ and $b\sin\phi$.

The circle described on the major axis of an ellipse as diameter, and with the same centre therefore as the ellipse, is known as the *Auxiliary Circle* of the ellipse.

The equation of the ellipse being $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, that of the auxiliary circle is $x^2+y^2=a^2$.

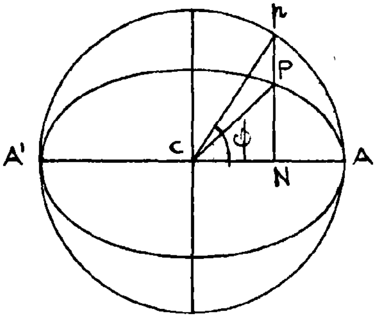
Draw an ellipse and its auxiliary circle. Let C be their common centre.

Let the ordinate of any point P on the ellipse cut the major axis in N and the auxiliary circle in p.

P and p are then called corresponding points on the ellipse and the auxiliary circle respectively.

Then, as CN, PN are the coordinates of a point P on the ellipse,

$$\therefore \ \frac{CN^2}{a^2}+\frac{PN^2}{b^2}=1.$$



And as CN, pN are the coordinates of the corresponding point p on the auxiliary circle,

$$\therefore \ \frac{CN^2}{a^2}+\frac{pN^2}{a^2}=1;$$

whence we have
$$\frac{PN^2}{pN^2}=\frac{b^2}{a^2},$$

or
$$\mathbf{PN : pN = b : a.}$$

The angle pCN between Cp and CA is called the *Eccentric Angle* of the point P on the ellipse. Denoting it by ϕ, the coordinates of p are seen to be $a\cos\phi$, $a\sin\phi$, and those of P are $a\cos\phi$, $b\sin\phi$.

Thus ϕ is the parameter of the point P, and the geometrical interpretation of the parameter in this case is that it is the eccentric angle of the point P.

3. Equations of the chord and tangent

Let P, Q be two points ϕ, ϕ' on the ellipse whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the line joining them is

$$\frac{x-a\cos\phi}{a(\cos\phi-\cos\phi')}=\frac{y-b\sin\phi}{b(\sin\phi-\sin\phi')},$$

i.e. $$b(x-a\cos\phi)2\cos\frac{\phi+\phi'}{2}\sin\frac{\phi-\phi'}{2}$$
$$=a(y-b\sin\phi)2\sin\frac{\phi+\phi'}{2}\sin\frac{\phi'-\phi}{2},$$

i.e. $$bx\cos\frac{\phi+\phi'}{2}+ay\sin\frac{\phi+\phi'}{2}$$
$$=ab\left(\cos\phi\cos\frac{\phi+\phi'}{2}+\sin\phi\sin\frac{\phi+\phi'}{2}\right)$$
$$=ab\cos\frac{\phi-\phi'}{2}.$$

Thus the equation of the chord joining the two points ϕ, ϕ' is

$$\mathbf{\frac{x}{a}\cos\frac{\varphi+\varphi'}{2}+\frac{y}{b}\sin\frac{\varphi+\varphi'}{2}=\cos\frac{\varphi-\varphi'}{2}}.$$

To get the equation of the tangent at the point ϕ it is only necessary to put $\phi'=\phi$ in the equation of the chord just obtained, and we have

$$\mathbf{\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=1}.$$

At a vertex of the ellipse ϕ is 0 or $\frac{\pi}{2}$, π or $\frac{3\pi}{2}$. Hence the tangents at the vertices are $x=\pm a$, $y=\pm b$.

If any diameter of an ellipse be drawn, i.e. a chord through its centre, the eccentric angles of its extremities clearly differ by π. Thus, if POP' be the diameter of an ellipse, and P is the point ϕ, P' will be the point $\pi+\phi$ and its coordinates $-a\cos\phi$, $-b\sin\phi$.

The equations of the tangents at P and P' are

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1 \quad \text{and} \quad \frac{x}{a}\cos(\pi+\phi)+\frac{y}{b}\sin(\pi+\phi)=1$$

respectively.

Their gradients are $-\frac{b}{a}\cot\phi$ and $-\frac{b}{a}\cot(\pi+\phi)$. As these are equal, we see that tangents at the extremities of any diameter of an ellipse are parallel and at the same perpendicular distance from the centre of the ellipse.

If we put $x'=a\cos\phi$, $y'=b\sin\phi$ in the equation of the tangent, it becomes

$$\mathbf{\frac{xx'}{a^2}+\frac{yy'}{b^2}=1,}$$

which is the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x',y').

This last form of the equation of the tangent to the ellipse can be obtained directly as follows:

Let (x',y'), (x'',y'') be two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then
$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1=\frac{x''^2}{a^2}+\frac{y''^2}{b^2}.$$

$$\therefore \quad \frac{(x'-x'')(x'+x'')}{a^2}=-\frac{(y'-y'')(y'+y'')}{b^2}. \tag{1}$$

The equation of the chord joining (x',y'), (x'',y'') is

$$\frac{x-x'}{x'-x''}=\frac{y-y'}{y'-y''}. \tag{2}$$

Combining equations (1) and (2),

$$\frac{(x-x')(x'+x'')}{a^2}+\frac{(y-y')(y'+y'')}{b^2}=0,$$

i.e.
$$\frac{x(x'+x'')}{a^2}+\frac{y(y'+y'')}{b^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{x'x''}{a^2}+\frac{y'y''}{b^2},$$

or, since
$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1,$$

$$\frac{x(x'+x'')}{a^2}+\frac{y(y'+y'')}{b^2}=1+\frac{x'x''}{a^2}+\frac{y'y''}{b^2}.$$

To obtain the equation of the tangent at x' put $x''=x'$ in this equation and we have

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{1}{2}\left(1+\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)=1.$$

The student familiar with the differential calculus can arrive at the equation of the tangent as follows:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \therefore\ \frac{y\,dy}{dx}=-\frac{b^2x}{a^2}.$$

Hence the gradient of the tangent at the point (x',y') is $-\frac{b^2}{a^2}\frac{x'}{y'}$, and its equation $y-y'=-\frac{b^2}{a^2}\frac{x'}{y'}(x-x')$, which reduces to

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1.$$

4. Equation of the normal

The normal at a point P on an ellipse is a line through P perpendicular to the tangent at P.

The equation of the ellipse being $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and P being the point $(a\cos\phi, b\sin\phi)$, the equation of the normal is

$$\frac{x}{b}\sin\phi-\frac{y}{a}\cos\phi=\frac{a\cos\phi\sin\phi}{b}-\frac{b\sin\phi\cos\phi}{a},$$

i.e.
$$\frac{\mathbf{ax}}{\mathbf{\cos\phi}}-\frac{\mathbf{by}}{\mathbf{\sin\phi}}=\mathbf{a^2-b^2=a^2e^2}.$$

If the point be (x',y'), the equation of the normal is

$$\frac{a^2x}{x'}-\frac{b^2y}{y'}=\frac{a^2x'}{x'}-\frac{b^2y'}{y'}=a^2-b^2.$$

5. Two tangents to an ellipse

The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point ϕ is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1.$$

Suppose this passes through a point (h,k), then

$$\frac{h}{a}\cos\phi+\frac{k}{b}\sin\phi=1.$$

In this equation put $\tan\frac{\phi}{2}=t$, then

$$\frac{h}{a}\cdot\frac{1-t^2}{1+t^2}+\frac{k}{b}\cdot\frac{2t}{1+t^2}=1,$$

i.e.

$$t^2\left(1+\frac{h}{a}\right)-2t\frac{k}{b}+1-\frac{h}{a}=0.$$

This equation being a quadratic in t has two roots, i.e. there are two values of t for every point (h,k), which means that two tangents can be drawn from a given point to an ellipse.

These tangents will be real, coincident, or imaginary according as the roots of the equation in t are real, equal, or unreal.

That is to say, according as

$$\frac{k^2}{b^2}\gtreqless\left(1+\frac{h}{a}\right)\left(1-\frac{h}{a}\right)\gtreqless 1-\frac{h^2}{a^2}$$

or

$$\frac{h^2}{a^2}+\frac{k^2}{b^2}\gtreqless 1,$$

which means, according as (h,k) is outside, on, or within the ellipse.

6. Tangential equation of an ellipse

To find the tangential equation of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we have to ascertain the condition that the line $lx+my+n=0$ should be a tangent to it.

Identifying $lx+my+n = 0$ with $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi = 1$, we have

$$\frac{\cos\phi}{al} = \frac{\sin\phi}{bm} = -\frac{1}{n};$$

whence

$$1 = \cos^2\phi+\sin^2\phi = \frac{a^2l^2+b^2m^2}{n^2}.$$

$$\therefore\ a^2l^2+b^2m^2 = n^2$$

is the required tangential equation of the ellipse.

By the same method of identification we can find the condition that the line $x\cos\alpha+y\sin\alpha = p$ or the line $y = mx+c$ should touch the ellipse.

Identifying $x\cos\alpha+y\sin\alpha = p$ with $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi = 1$, we have

$$p = \frac{b\sin\alpha}{\sin\phi} = \frac{a\cos\alpha}{\cos\phi}.$$

$$\therefore\ a^2\cos^2\alpha+b^2\sin^2\alpha = p^2(\cos^2\phi+\sin^2\phi) = p^2.$$

In the second case

$$\frac{\cos\phi}{am} = -\frac{\sin\phi}{b} = -\frac{1}{c}.$$

$$\therefore\ 1 = \cos^2\phi+\sin^2\phi = \frac{a^2m^2+b^2}{c^2}.$$

$$\therefore\ c = \pm\sqrt{(a^2m^2+b^2)}.$$

Hence, for any particular gradient m, i.e. any specified direction, two parallel tangents can be drawn to the ellipse whose equations are

$$y = mx+\sqrt{(a^2m^2+b^2)} \quad\text{and}\quad y = mx-\sqrt{(a^2m^2+b^2)}.$$

7. Pole and polar with regard to an ellipse

The point of intersection of the tangents at the extremities of a chord of an ellipse is called the *Pole* of that chord with regard to the ellipse, and the chord is called the *Polar* of that point.

We shall now find the coordinates of the point at which the tangents at two points ϕ, ϕ' on the ellipse intersect.

The equations of the tangents at ϕ and ϕ' being

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1 \quad\text{and}\quad \frac{x}{a}\cos\phi'+\frac{y}{b}\sin\phi'=1,$$

where they intersect

$$\frac{x/a}{\sin\phi-\sin\phi'}=\frac{y/b}{\cos\phi'-\cos\phi}=\frac{-1}{\sin\phi'\cos\phi-\sin\phi\cos\phi'}.$$

$$\therefore\ \frac{x}{a}=\frac{\sin\phi-\sin\phi'}{\sin(\phi-\phi')}=\frac{2\cos\frac{\phi+\phi'}{2}\sin\frac{\phi-\phi'}{2}}{2\sin\frac{\phi-\phi'}{2}\cos\frac{\phi-\phi'}{2}}=\frac{\cos\frac{\phi+\phi'}{2}}{\cos\frac{\phi-\phi'}{2}}$$

and

$$\frac{y}{b}=\frac{\cos\phi'-\cos\phi}{\sin(\phi-\phi')}=\frac{2\sin\frac{\phi-\phi'}{2}\sin\frac{\phi+\phi'}{2}}{2\sin\frac{\phi-\phi'}{2}\cos\frac{\phi-\phi'}{2}}=\frac{\sin\frac{\phi+\phi'}{2}}{\cos\frac{\phi-\phi'}{2}},$$

i.e. the tangents at two points ϕ, ϕ' of an ellipse intersect at the point whose coordinates are

$$\frac{\mathbf{a}\cos\frac{\boldsymbol{\varphi}+\boldsymbol{\varphi}'}{\mathbf{2}}}{\cos\frac{\boldsymbol{\varphi}-\boldsymbol{\varphi}'}{\mathbf{2}}},\quad \frac{\mathbf{b}\sin\frac{\boldsymbol{\varphi}+\boldsymbol{\varphi}'}{\mathbf{2}}}{\cos\frac{\boldsymbol{\varphi}-\boldsymbol{\varphi}'}{\mathbf{2}}}.$$

Let (h, k) be the coordinates of a point whose polar we desire to find with regard to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Suppose the tangents from (h, k) to the ellipse touch it at the points ϕ and ϕ'.

Then $$h=\frac{a\cos\frac{\phi+\phi'}{2}}{\cos\frac{\phi-\phi'}{2}},\qquad k=\frac{b\sin\frac{\phi+\phi'}{2}}{\cos\frac{\phi-\phi'}{2}}.$$

The chord joining these points has for its equation

$$\frac{x}{a}\cos\frac{\phi+\phi'}{2}+\frac{y}{b}\sin\frac{\phi+\phi'}{2}=\cos\frac{\phi-\phi'}{2},$$

i.e. $$\frac{xh}{a^2}+\frac{yk}{b^2}=1,$$

which is the equation of the polar of (h, k) with regard to the ellipse.

We note that if h and k are real, this line $\frac{xh}{a^2}+\frac{yk}{b^2}=1$ must also be real.

Thus every point has a real polar no matter whether the tangents from it to the ellipse are real or imaginary. This is equivalent to saying that the polar will be a real line whether the point is within or without the ellipse. If the point (h, k) is on the ellipse, the equation of its polar is identical with that of the tangent at (h, k). Hence the polar of a point on an ellipse is the tangent at that point.

If the polar of a point (h, k) with regard to an ellipse passes through another point (H, K), then the polar of (H, K) will pass through the point (h, k).

The method of proof is the same as that employed in the case of the circle and the parabola.

The equation of the ellipse being $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, that of the polar of (h, k) with regard to it is $\frac{xh}{a^2}+\frac{yk}{b^2}=1$.

If this line passes through the point (H, K),

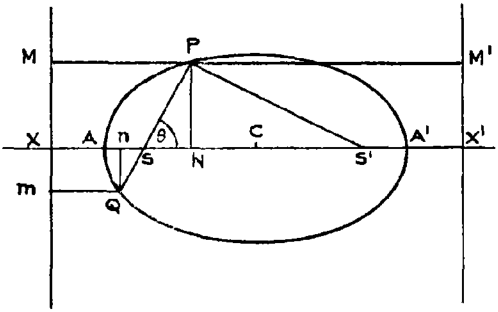
$$\frac{Hh}{a^2}+\frac{Kk}{b^2}=1. \tag{1}$$

The polar of (H, K) with regard to the ellipse is

$$\frac{xH}{a^2}+\frac{yK}{b^2}=1.$$

Equation (1) shows that $x = h$, $y = k$ satisfy this equation.
$\therefore$ the polar of (H, K) passes through (h, k).

8. Polar equation of an ellipse



Let P be a point on an ellipse whose semi-latus rectum is l, and PSQ a focal chord making an angle θ with the major axis.

Draw PM, Qm perpendicular to the directrix and PN, Qn perpendicular to the major axis.

The polar coordinates of P are $r = SP$ and $\theta = \angle PSN$.

In the diagram

$$r = SP = ePM = eNX = e(XS+SN)$$

$$= e\left(\frac{l}{e}+r\cos\theta\right).$$

$$\therefore \quad \mathbf{\frac{l}{r} = 1-e\cos\theta}$$

is the polar equation of the ellipse.

If θ were measured in a clockwise direction with XSX' as initial line, the polar equation of the ellipse would be

$$\frac{l}{r} = 1+e\cos\theta.$$

From the equation first obtained

$$\frac{l}{SP} = 1-e\cos\theta$$

and $$\frac{l}{SQ} = 1 - e\cos(\pi+\theta) = 1 + e\cos\theta.$$

$$\therefore\ \frac{1}{SP} + \frac{1}{SQ} = \frac{2}{l}.$$

Showing that the semi-latus rectum is a harmonic mean between the segments of a focal chord of an ellipse.

If MP be produced to cut the other directrix of the ellipse in M', then

$$SP + S'P = e(PM + PM') = eMM' = eXX' = 2eCX$$

$$= 2e.\frac{a}{e} = 2a.$$

Hence the sum of the focal radii of any point on an ellipse is always equal to the length of its major axis.

The polar equation of the tangent at a point on the ellipse $\frac{l}{r} = 1 - e\cos\theta$ whose focal radius makes an angle α with the major axis can be found thus:

Suppose that the line

$$\frac{l}{r} = A\cos\theta + B\sin\theta$$

is the tangent at the point.

Then $$1 - e\cos\alpha = A\cos\alpha + B\sin\alpha.$$

Put $\tan\frac{\alpha}{2} = t$, and this becomes

$$1 = (A+e)\frac{1-t^2}{1+t^2} + B\frac{2t}{1+t^2},$$

i.e. $$t^2(1+A+e) - 2Bt + (1-A-e) = 0,$$

of which both roots must be $\tan\frac{\alpha}{2}$.

$$\therefore\ \tan^2\frac{\alpha}{2}=\frac{1-A-e}{1+A+e}$$

and
$$\cos\alpha=\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=A+e,$$

also
$$2\tan\frac{\alpha}{2}=\frac{2B}{1+A+e}=\frac{2B}{1+\cos\alpha}.$$

$$\therefore\ B=2\cos^2\frac{\alpha}{2}\tan\frac{\alpha}{2}=\sin\alpha.$$

Hence the equation of the tangent at the point $\frac{l}{1-e\cos\alpha}$, α is

$$\frac{l}{r}=\cos\theta(\cos\alpha-e)+\sin\alpha\sin\theta.$$

$$\therefore\ \boldsymbol{\frac{1}{r}=\cos(\theta-\alpha)-e\cos\theta.}$$

The polar equation of the normal at the point $\frac{l}{1-e\cos\alpha}$, α must be of the form $\frac{K}{r}=\cos\left(\frac{\pi}{2}+\theta-\alpha\right)-e\cos\left(\frac{\pi}{2}+\theta\right)$, which is perpendicular to the line $\frac{l}{r}=\cos(\theta-\alpha)-e\cos\theta$ for all values of K.

If this line passes through the point $\frac{l}{1-e\cos\alpha}$, α, then

$$\frac{K(1-e\cos\alpha)}{l}=e\sin\alpha.$$

$\therefore$ the equation of the normal at the point $\frac{l}{1-e\cos\alpha}$, α is

$$-\frac{l}{r}\cdot\frac{e\sin\alpha}{1-e\cos\alpha}=\sin(\theta-\alpha)-e\sin\theta.$$

Where the tangents at two points α, β intersect, we have

$$\frac{l}{r} = \cos(\theta-\alpha)-e\cos\theta$$

and $$\frac{l}{r} = \cos(\theta-\beta)-e\cos\theta.$$

$$\therefore\ \cos(\theta-\alpha) = \cos(\theta-\beta);$$

whence $$\theta-\alpha = \beta-\theta \text{ and } \theta = \frac{\alpha+\beta}{2}.$$

Hence, if TP, TQ be two tangents to an ellipse and SA be the major axis,

$$\angle TSP = \angle PSA-\angle TSA = \alpha-\frac{\alpha+\beta}{2} = \frac{\alpha-\beta}{2}$$

and $$\angle TSQ = \angle TSA-\angle QSA = \frac{\alpha+\beta}{2}-\beta = \frac{\alpha-\beta}{2},$$

showing that $\angle TSP = \angle TSQ$.

$\therefore$ any two tangents to an ellipse subtend equal angles at a focus.

The tangents at the extremities of a focal chord meet on the directrix.

The extremities of the focal chord have as tangents the lines whose equations are

$$\frac{l}{r} = \cos(\theta-\alpha)-e\cos\theta$$

and $$\frac{l}{r} = \cos(\theta-\pi-\alpha)-e\cos\theta = -\cos(\theta-\alpha)-e\cos\theta.$$

Where they intersect we have, by addition,

$$\frac{l}{r} = -e\cos\theta,$$

which is the polar equation of the directrix.

Suppose the two tangents at the extremities of the focal chord PSQ intersect at a point T on the directrix.

We have proved that $\angle TSP = \angle TSQ$.

$\therefore$ TS must be perpendicular to PSQ.

9. Various properties of an ellipse

We will suppose the equation of the ellipse to be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

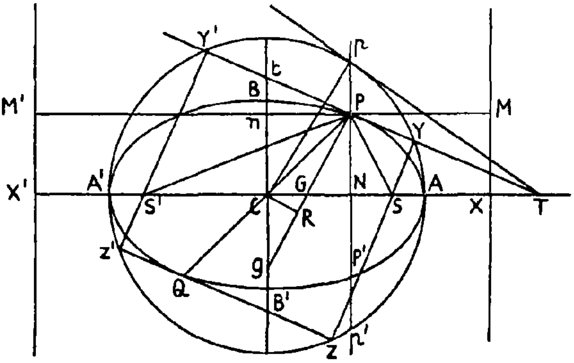
Let S, S' be its foci, ACA' and BCB' its axes, XM and $X'M'$ its directrices.

Let P be any point ϕ on the ellipse. Draw PN perpendicular to the major axis and produce it to cut the ellipse again in P' and the auxiliary circle in p and p'.

Through P draw MPM' parallel to the major axis to cut the directrices in M and M' and the minor axis in n.

Let the tangent at P to the ellipse cut the major axis in T, and the minor axis in t.

Let the normal at P cut the major axis in G and the minor axis in g.



Draw lines through the foci S and S' perpendicular to the tangents at P and Q, where PCQ is a diameter of the ellipse and to meet them in Y, Y', Z, Z' respectively. Join SP, $S'P$, Cp, and draw CR perpendicular to PGg.

The following theorems will now be proved.

(1) The tangents at any pair of corresponding points P, p on the ellipse and the auxiliary circle meet at a point on the major axis.

The equation of the tangent at P, the point ϕ, to the ellipse is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi = 1.$$

Where this cuts $y = 0$ we have $x = a\sec\phi$.

The equation of the tangent at p, the point $(a\cos\phi, a\sin\phi)$, to the circle $x^2+y^2 = a^2$ is

$$\frac{x}{a}\cos\phi+\frac{y}{a}\sin\phi = 1,$$

and where it cuts $y = 0$, $x = a\sec\phi$.

Hence both lines meet at the point T on the axis, and

$$CT = a\sec\phi.$$

(2) If the tangent at P meets the axes in T and t, then $CN.CT = a^2$ and $Cn.Ct = b^2$.

We saw in (1) that $CT = a\sec\phi$ and $CN = a\cos\phi$.

$$\therefore\ CN.CT = a^2.$$

Where the tangent at P cuts the minor axis, we have, putting $x = 0$ in the equation $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi = 1$,

$$Ct = y = b\operatorname{cosec}\phi$$

and
$$Cn = PN = b\sin\phi.$$

$$\therefore\ Cn.Ct = b^2.$$

(3) To prove that $CG = e^2CN$ and $Cg = -\frac{a^2e^2}{b^2}Cn$.

The equation of PG, the normal at the point ϕ, is

$$\frac{ax}{\cos\phi}-\frac{by}{\sin\phi} = a^2-b^2.$$

Where this cuts the major axis $y = 0$, we have

$$a.CG = (a^2-b^2)\cos\phi = a^2e^2\cos\phi.$$

$$\therefore\ CG = e^2.a\cos\phi = e^2CN.$$

Where the normal cuts the minor axis $x = 0$, we have

$$Cg = -\frac{a^2-b^2}{b}\sin\phi = -\frac{a^2e^2}{b^2}.b\sin\phi$$

$$= -\frac{a^2e^2}{b^2}.Cn.$$

(4) To prove that $SG = eSP$.

In the diagram

$$SP = ePM = eNX = e(CX-CN)$$

$$= e\left(\frac{a}{e}-a\cos\phi\right) = a(1-e\cos\phi).$$

Also $$SG = CS-CG = CS-e^2CN$$

$$= ae-e^2a\cos\phi$$

$$= ae(1-e\cos\phi)$$

$$= e.SP.$$

(5) To prove that the normal bisects the angle between the focal radii of a point on an ellipse and that the tangent makes equal angles with them.

We have just proved that $SG = eSP$, and from symmetry $S'G = eS'P$, $\quad\therefore\ SP : S'P = SG : S'G.$

$\therefore$ PG is the bisector of $\angle SPS'$.

And as the tangent PT is perpendicular to PG, it must be the external bisector of $\angle SPS'$, and therefore SP and $S'P$ will be equally inclined to it.

(6) To prove that $PG : Pg = NG : NC = b^2 : a^2$.

From the diagram
$$\frac{Pg}{PG}=\frac{Pn}{NG}$$
$$=\frac{NC}{NG}.$$

Also
$$\frac{GN}{CN}=\frac{CN-CG}{CN}=\frac{CN-e^2CN}{CN}=1-e^2=\frac{b^2}{a^2}.$$

(7) To prove that $PG.PR=b^2$ and $Pg.PR=a^2$.

PR is the length of the perpendicular from the centre C to the tangent at P, i.e. from the point $(0,0)$ to the line
$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi-1=0.$$
$$\therefore\ PR=\frac{1}{\sqrt{\left(\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}\right)}}=\frac{ab}{\sqrt{(a^2\sin^2\phi+b^2\cos^2\phi)}}.$$

Also $\quad PG^2=PN^2+NG^2=b^2\sin^2\phi+\frac{b^4}{a^2}\cos^2\phi.$
$$\therefore\ PG=\frac{b}{a}\sqrt{(a^2\sin^2\phi+b^2\cos^2\phi)}.$$
$$\therefore\ PR.PG=b^2.$$

And as $Pg=\frac{a^2}{b^2}PG$, $\quad\therefore\ PR.Pg=a^2$.

(8) To prove that Y and Y' lie on the auxiliary circle, that $SY.S'Y'=b^2$, and that $SY.SZ=S'Y'.S'Z'$.

The equation of the tangent at P, the point ϕ, is
$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1.$$

That of a line through $(ae,0)$ perpendicular to it
$$\frac{x}{b}\sin\phi-\frac{y}{a}\cos\phi=\frac{ae\sin\phi}{b}.$$

Squaring and adding these,

$$(x^2+y^2)\left(\frac{a^2\sin^2\phi+b^2\cos^2\phi}{a^2b^2}\right) = 1+\frac{a^2e^2\sin^2\phi}{b^2}$$

$$= \frac{a^2\sin^2\phi+b^2\cos^2\phi}{b^2}.$$

$$\therefore\ x^2+y^2 = a^2,$$

i.e. the point Y lies on the auxiliary circle, and, from symmetry, Y' must also do so.

The perpendicular from S $(ae, 0)$ on the tangent at P is

$$\frac{e\cos\phi-1}{\sqrt{\left(\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}\right)}} = \frac{(e\cos\phi-1)ab}{\sqrt{(a^2\sin^2\phi+b^2\cos^2\phi)}},$$

and that from S' $(-ae, 0)$ is of length $\dfrac{(-e\cos\phi-1)ab}{\sqrt{(a^2\sin^2\phi+b^2\cos^2\phi)}}$.

$$\therefore\ SY.S'Y' = \frac{1-e^2\cos^2\phi}{a^2\sin^2\phi+b^2\cos^2\phi}a^2b^2.$$

But

$$a^2\sin^2\phi+b^2\cos^2\phi = a^2\sin^2\phi+a^2(1-e^2)\cos^2\phi = a^2(1-e^2\cos^2\phi).$$

$$\therefore\ SY.S'Y' = b^2.$$

Again, as Y, Y', Z, Z' all lie on the auxiliary circle,

$$\therefore\ SY.SZ = SA.SA' = a(1-e).a(1+e) = a^2(1-e^2) = b^2$$

and $$S'Y'.S'Z' = S'A'.S'A = SA.SA' = b^2.$$

(9) If PSQ is a focal chord the tangents at its extremities meet on the directrix. If they meet at a point T, then TS is perpendicular to PQ.

These two theorems have already been proved in Art. 8 by the use of polar coordinates. They can also be proved thus:

Let the extremities of the focal chord PQ be the points θ, ϕ on the ellipse.

The equation of the chord joining them is

$$\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2},$$

and as this passes through the focus $(ae, 0)$, we have

$$e\cos\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}.$$

The abscissa of T we know to be $\dfrac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}=\dfrac{a}{e}$.

Hence the point lies on the directrix whose equation is $x=\dfrac{a}{e}$.

The ordinate of T is

$$\frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}=\frac{b}{e}\tan\frac{\theta+\phi}{2}.$$

The equation of TS being

$$\frac{x-ae}{ae-\frac{a}{e}}=\frac{y}{-\frac{b}{e}\tan\frac{\theta+\phi}{2}},$$

the gradient of TS is accordingly $\dfrac{b\tan\frac{\theta+\phi}{2}}{a(1-e^2)}$, which is equal to $\dfrac{a}{b}\tan\dfrac{\theta+\phi}{2}$.

And the gradient of PQ being $-\dfrac{b}{a}\cot\dfrac{\theta+\phi}{2}$, we see that the product of the gradients is -1; hence TS is perpendicular to PQ.

(10) The locus of the point of intersection of two perpendicular tangents to an ellipse is a circle of radius $\sqrt{(a^2+b^2)}$.

The equation of any tangent to an ellipse being

$$y = mx+\sqrt{(a^2m^2+b^2)},$$

that of a perpendicular tangent, whose gradient is $-\frac{1}{m}$, will be

$$y = -\frac{1}{m}x+\sqrt{\left(\frac{a^2}{m^2}+b^2\right)},$$

i.e. $$my+x = \sqrt{(a^2+b^2m^2)}.$$

$$\therefore\ (y-mx)^2+(my+x)^2 = a^2m^2+b^2+a^2+b^2m^2.$$

$$\therefore\ (x^2+y^2)(1+m^2) = (a^2+b^2)(1+m^2);$$

whence $$\mathbf{x^2+y^2 = a^2+b^2}$$

is the required locus.

This circle is known as the *Director Circle*; in the case of the parabola, it became the directrix itself, which may be regarded as part of the circumference of a circle of infinite radius whose centre is at infinity on the axis.

The equation of the director circle can also be found thus:

If (X, Y) is on the tangent to the ellipse at a point ϕ, then

$$\frac{X}{a}\cos\phi+\frac{Y}{b}\sin\phi = 1.$$

Putting $\tan\phi = t$, $$\left(\frac{X}{a}+\frac{Y}{b}t\right)^2 = 1+t^2,$$

i.e. $$t^2\left(\frac{Y^2}{b^2}-1\right)+\frac{2XYt}{ab}+\frac{X^2}{a^2}-1 = 0.$$

But if the tangents are perpendicular, the product of the roots of this equation $= -\frac{b^2}{a^2}$.

$$\therefore \quad \frac{X^2}{a^2}-1 = -\frac{b^2}{a^2}\left(\frac{Y^2}{b^2}-1\right),$$

i.e. $$X^2+Y^2 = a^2+b^2.$$

(11) The centre of an ellipse which always touches two fixed perpendicular straight lines, moves on a circle of radius $\sqrt{(a^2+b^2)}$.

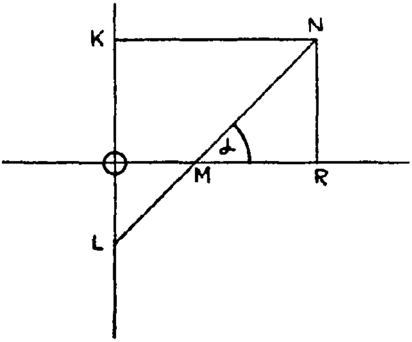
This is really the converse of the previous theorem. The centre of the given ellipse being C and the point where the perpendicular lines meet being O, we have $CO^2 = a^2+b^2$.

So if O be fixed, C clearly moves on a circle with O as its centre and radius $\sqrt{(a^2+b^2)}$, as it slides between the two perpendicular straight lines.

10. Methods of drawing an ellipse

(1) Take a circle whose diameter is the major axis of the required ellipse. Draw ordinates from points of this circle to the diameter and divide them all so that the part of the ordinate nearest to the diameter is to the whole ordinate as $b : a$, $2a$ and $2b$ being the major and minor axes of the ellipse.

(2) Take a straight line LMN such that $LN = a$ and $MN = b$. Let the line LMN move so that L and M are always on two fixed lines Oy and Ox at right angles. Then N will trace out an ellipse whose major and minor axes are $2a$ and $2b$ respectively.



The proof of this is as follows:

Let LMN be one position of the line, L being on Oy and M on Ox. Draw NR, NK perpendicular to Ox and Oy respectively, and let $\angle NMR = \alpha$.

Let (x, y) be the coordinates of N.

Then

$$x = OR = NK = LN \cos LNK = a \cos \alpha$$

and
$$y = NR = MN \sin \angle NMR = b \sin \alpha.$$

$$\therefore \ \frac{x^2}{a^2}+\frac{y^2}{b^2} = \cos^2\alpha+\sin^2\alpha = 1,$$

which shows that the locus of N is an ellipse whose centre is at the origin and axes of lengths $2a$, $2b$.

(3) It has been proved that the sum of the focal radii of any point on an ellipse is always equal to the major axis of the ellipse.

If the major axis of an ellipse be $2a$ and its eccentricity e, choose two points on the paper which are distant $2ae$ from each other.

Now take a piece of string or cotton of length $2a$ and fasten its ends at the two fixed points. Keeping the string taut, mark with a pencil the vertices of the various triangles which can be formed in different positions of the string.

These vertices will all lie on an ellipse which can be traced by letting the pencil run in the angle formed by the two segments of the string when it is kept taut. Its foci will be the two fixed points where the ends of the string were fastened and its centre the middle point of the line joining them.

11. To find the coordinates of the centre of a conic represented by the general equation of the second degree

The equation of the conic being

$$ax^2+2hxy+by^2+2gx+2fy+c = 0,$$

suppose that its centre is the point (x', y').

Refer to parallel axes through (x', y') and we have

$$a(x+x')^2+b(y+y')^2+2h(x+x')(y+y')$$
$$+2g(x+x')+2f(y+y')+c = 0. \quad (1)$$

Now the equation of any conic referred to perpendicular axes through its centre would be of the form

$$Ax^2+By^2+2Hxy = 1,$$

since when we put $-x$ for x and $-y$ for y in this equation it remains unaltered in form.

This equation has no x term and no y term. Hence in the equation (1) the coefficients of x and y must each be zero.

$$\therefore\ ax'+hy'+g = 0$$

and $$hx'+by'+f = 0,$$

whence $$\frac{x'}{hf-bg} = \frac{y'}{gh-af} = \frac{1}{ab-h^2},$$

which give the coordinates of the centre of the conic.

The student familiar with the calculus will see that the equations determining the centre can be obtained by differentiating the general equation with regard to x regarding y as a constant, and then differentiating it with regard to y regarding x as a constant.

12. Typical examples

✓Ex. 1. Find the condition that the line $lx+my+n = 0$ should be a normal of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$

Identifying $$lx+my+n = 0$$

with $$\frac{ax}{\cos\phi}-\frac{by}{\sin\phi} = a^2-b^2 = a^2e^2,$$

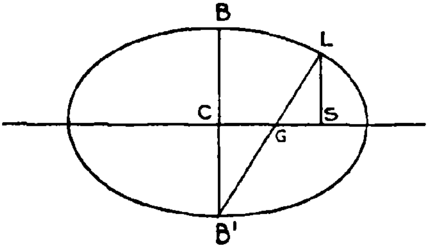
we have $$\frac{l\cos\phi}{a} = -\frac{m\sin\phi}{b} = -\frac{n}{a^2e^2}.$$

$$\therefore\ 1 = \cos^2\phi+\sin^2\phi = \frac{n^2}{a^4e^4}\left(\frac{a^2}{l^2}+\frac{b^2}{m^2}\right),$$

i.e.
$$\frac{a^2}{l^2}+\frac{b^2}{m^2} = \frac{(a^2-b^2)^2}{n^2}.$$

Ex. 2. The normal at an extremity of the latus rectum of an ellipse passes through an extremity of the minor axis. Find an equation giving the eccentricities of the ellipses for which this is possible.

If the normal at the point $[ae, a(1-e^2)]$ cuts the major axis in G, we have $CG = e^2.ae = ae^3$.



Now
$$\frac{CB'}{CG} = \frac{LS}{SG} = \frac{LS}{CS-CG} = \frac{\frac{b^2}{a}}{ae-ae^3} = \frac{1}{e}.$$

$$\therefore\ \frac{1}{e^2} = \frac{b^2}{CG^2} = \frac{b^2}{a^2e^6} = \frac{1-e^2}{e^6}.$$

$$\therefore\ e^4+e^2-1 = 0$$

is the required equation.

Ex. 3. Two tangents are drawn to an ellipse whose gradients are m_1 and m_2. Find the locus of their point of intersection (1) if m_1m_2 is constant, (2) if $\frac{1}{m_1}+\frac{1}{m_2}$ is constant.

Suppose that ϕ_1, ϕ_2 are the points of contact of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

The equation of the tangent at ϕ being $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$,

$$m_1=-\frac{b}{a}\cot\phi_1 \quad \text{and} \quad m_2=-\frac{b}{a}\cot\phi_2.$$

Let $t=\tan\phi$. The equation of the tangent can then be written

$$\frac{x}{a}+\frac{y}{b}t=\sqrt{(1+t^2)},$$

then $$t^2\left(\frac{y^2}{b^2}-1\right)+\frac{2xy}{ab}t+\frac{x^2}{a^2}-1=0,$$

of which equation the roots will be $\tan\phi_1$ and $\tan\phi_2$.

Thus $$\tan\phi_1+\tan\phi_2=-\frac{2xyb}{a(y^2-b^2)}$$

and $$\tan\phi_1\tan\phi_2=\frac{b^2(x^2-a^2)}{a^2(y^2-b^2)}.$$

If $m_1m_2=\kappa$, we have

$$\frac{b^2}{a^2}\cot\phi_1\cot\phi_2=\kappa.$$

$$\therefore\ \frac{b^2}{a^2}=\frac{\kappa b^2(x^2-a^2)}{a^2(y^2-b^2)},$$

whence the locus is $\kappa(x^2-a^2)=y^2-b^2$.

If $\frac{1}{m_1}+\frac{1}{m_2}=\lambda$, we have

$$-\frac{b}{a}\lambda=\tan\phi_1+\tan\phi_2=-\frac{2xyb}{a(y^2-b^2)},$$

whence $$2xy=\lambda(y^2-b^2),$$

which is the required locus in the second case.

✓Ex. 4. Find the locus of the pole of a normal chord of an ellipse.

Suppose the normal at the point ϕ meets the ellipse again at the point ϕ'.

Then $$\frac{x}{a}\cos\frac{\phi+\phi'}{2}+\frac{y}{b}\sin\frac{\phi+\phi'}{2}=\cos\frac{\phi-\phi'}{2}$$

must be identical with

$$\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2=a^2e^2.$$

$$\therefore\ \frac{\cos\frac{\phi+\phi'}{2}}{a^2\sec\phi}=-\frac{\sin\frac{\phi+\phi'}{2}}{b^2\operatorname{cosec}\phi}=\frac{\cos\frac{\phi-\phi'}{2}}{a^2e^2}.$$

If (X, Y) be the point where the tangents at ϕ and ϕ' meet,

$$X=\frac{a\cos\frac{\phi+\phi'}{2}}{\cos\frac{\phi-\phi'}{2}}=\frac{a}{e^2\cos\phi}$$

and $$Y=-\frac{b\sin\frac{\phi+\phi'}{2}}{\cos\frac{\phi-\phi'}{2}}=-\frac{b^3}{a^2e^2\sin\phi}.$$

$$\therefore\ 1=\cos^2\phi+\sin^2\phi=\frac{a^2}{e^4X^2}+\frac{b^6}{a^4e^4Y^2}.$$

$$\therefore\ a^4e^4X^2Y^2=a^6Y^2+b^6X^2,$$

which is the equation of the locus required.

✓Ex. 5. Show that the equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose middle point is (h, k), is

$$hb^2(x-h)+ka^2(y-k)=0.$$

The equation of any line through (h, k) may be written as

$$y-k=m(x-h).$$

Where this line cuts the ellipse, we have

$$b^2x^2+a^2(mx-mh+k)^2=a^2b^2.$$

If the roots of this equation are x_1 and x_2, then

$$2h = x_1+x_2 = \frac{2ma^2(mh-k)}{b^2+a^2m^2}.$$

$$\therefore\ m = -\frac{hb^2}{ka^2},$$

and the equation of the chord is

$$hb^2(x-h)+ka^2(y-k) = 0.$$

Ex. 6. Find the polar equation of the chord joining two points on an ellipse whose vectorial angles are α and β.

The equation of the ellipse being $\frac{l}{r} = 1-e\cos\theta$, let that of the chord be $\frac{l}{r} = A\cos\theta + B\sin\theta$.

Then $$1-e\cos\alpha = A\cos\alpha + B\sin\alpha$$
and $$1-e\cos\beta = A\cos\beta + B\sin\beta.$$
$$\therefore\ \cos\beta-\cos\alpha = B\sin(\alpha-\beta)$$
and $$B = \frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}.$$

Also $$\sin\beta-\sin\alpha + e\sin(\alpha-\beta) = -A\sin(\alpha-\beta),$$
i.e. $$(A+e)\cos\frac{\alpha-\beta}{2} = \cos\frac{\alpha+\beta}{2}.$$

$\therefore$ the equation of the chord is

$$\frac{l}{r} = \left(\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} - e\right)\cos\theta + \frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\sin\theta,$$

i.e. $$\frac{l}{r}\cos\frac{\alpha-\beta}{2} = \cos\left(\theta-\frac{\alpha+\beta}{2}\right) - e\cos\frac{\alpha-\beta}{2}\cos\theta.$$

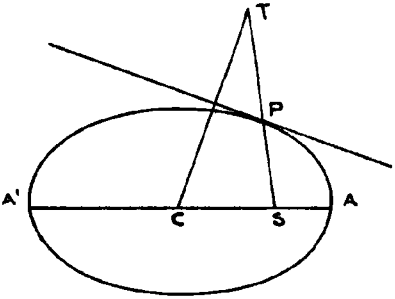
N.B. If we put $\alpha = \beta$ in this equation we get the equation of the tangent at the point whose vectorial angle is α, viz.

$$\frac{l}{r} = \cos(\theta-\alpha)-e\cos\theta,$$

which is an alternative way of arriving at this result to that employed in Art. 8.

Ex. 7. The centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is at C, and S is one of the foci. If the line through C perpendicular to the tangent at any point P on the ellipse cuts SP (produced if necessary) at T, show that the locus of T is a circle whose centre is S. [J.]

Let P be any point ϕ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.



The equation of the tangent at ϕ is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi = 1.$$

That of the line through $(0, 0)$ perpendicular to it is

$$ax\sin\phi - by\cos\phi = 0. \qquad (1)$$

The equation of SP is

$$\frac{x-ae}{a(e-\cos\phi)} = -\frac{y}{b\sin\phi},$$

i.e.

$$b(x-ae)\sin\phi - ay\cos\phi + aey = 0. \qquad (2)$$

Solving equations (1) and (2) for $\sin\phi$ and $\cos\phi$, we have

$$\frac{\sin\phi}{-abey^2} = \frac{\cos\phi}{-a^2exy} = \frac{1}{-a^2xy+b^2y(x-ae)}.$$

$$\therefore\ 1 = \sin^2\phi+\cos^2\phi$$

$$= \frac{a^2b^2e^2y^2}{\{b^2(x-ae)-a^2x\}^2} + \frac{a^4e^2x^2}{\{b^2(x-ae)-a^2x\}^2},$$

but

$$b^2(x-ae)-a^2x = -ab^2e-x(a^2-b^2) = -ab^2e-a^2e^2x$$

$$= -ae(b^2+xae).$$

$$\therefore\ (b^2+xae)^2 = b^2y^2+a^2x^2,$$

i.e.

$$x^2+y^2-2aex = b^2,$$

which is the equation of a circle whose centre is S $(ae, 0)$.

Ex. 8. The normal at P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ meets the radius of the auxiliary circle to the corresponding point P' in R. Prove that the locus of R is the circle

$$x^2+y^2 = (a+b)^2. \qquad \text{[W.]}$$

Let P be the point ϕ. The normal at P is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2e^2,$$

and the equation of CP' is $x = y\cot\phi$.

In the equation of the normal, rewritten

$$ax-by\cot\phi = \frac{a^2e^2}{\sqrt{(1+\tan^2\phi)}},$$

put $\cot\phi = \frac{x}{y}$, and we have

$$(a-b)x = \frac{a^2e^2x}{\sqrt{(x^2+y^2)}} = \frac{(a^2-b^2)x}{\sqrt{(x^2+y^2)}},$$

i.e.

$$x^2+y^2 = (a+b)^2.$$

Ex. 9. A chord PQ of an ellipse, centre C, is drawn in a fixed direction and meets the minor axis in Z. W is a point on the major axis equidistant from P and Q; show that $\dfrac{CW}{CZ}$ is constant. [L.]

Let P and Q be the points ϕ_1, ϕ_2 on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The gradient of PQ, i.e. $-\dfrac{a}{b}\cot\dfrac{\phi_1+\phi_2}{2}$, is constant.

$\therefore$ $\phi_1+\phi_2$ is constant.

Let W be the point $(\lambda, 0)$ on the major axis. Then

$$(\lambda-a\cos\phi_1)^2+b^2\sin^2\phi_1 = (\lambda-a\cos\phi_2)^2+b^2\sin^2\phi_2.$$

$$\therefore\ 2\lambda a(\cos\phi_2-\cos\phi_1)+a^2(\cos^2\phi_1-\cos^2\phi_2)$$
$$-b^2(\cos^2\phi_1-\cos^2\phi_2) = 0.$$

$$\therefore\ 2\lambda a = (a^2-b^2)(\cos\phi_2+\cos\phi_1).$$

CZ, the ordinate of the point in which PQ cuts the minor axis, is obtained by putting $x = 0$ in the equation of the chord.

Hence
$$\frac{CZ}{b}\sin\frac{\phi_1+\phi_2}{2} = \cos\frac{\phi_1-\phi_2}{2}.$$

$$\therefore\ \frac{CW}{CZ} = \frac{(a^2-b^2)(\cos\phi_1+\cos\phi_2)}{2a}\,\frac{\sin\dfrac{\phi_1+\phi_2}{2}}{b\cos\dfrac{\phi_1-\phi_2}{2}}$$

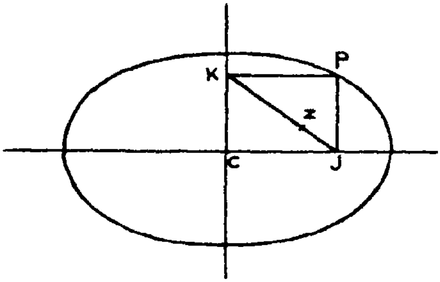
$$= \frac{(a^2-b^2)\sin\dfrac{\phi_1+\phi_2}{2}\,.\cos\dfrac{\phi_1+\phi_2}{2}}{ab}$$

$$= \frac{a^2-b^2}{2ab}\sin(\phi_1+\phi_2).$$

And as $\phi_1+\phi_2$ is constant, $\therefore\ \dfrac{CW}{CZ} =$ a constant.

Ex. 10. P is a point on the ellipse $b^2x^2+a^2y^2=a^2b^2$ and J and K are the feet of the perpendiculars from P to the major and minor axes. On the line JK a point Z is taken dividing JK in the ratio $l : m$. Find the locus of Z.

Further, find the value of the ratio $l:m$ when ZJ is the normal to the locus of Z. [B.]



Let P be the point ϕ. Then J is the point $(a\cos\phi, 0)$ and K the point $(0, b\sin\phi)$.

If Z divides JK in the ratio $l:m$, the coordinates (X, Y) of Z are

$$X = \frac{ma\cos\phi}{l+m}, \qquad Y = \frac{lb\sin\phi}{l+m}.$$

$$\therefore \quad \frac{(l+m)^2X^2}{m^2a^2} + \frac{(l+m)^2Y^2}{l^2b^2} = \cos^2\phi + \sin^2\phi = 1.$$

The equation of the normal to this ellipse at the point Z is

$$\frac{x - \dfrac{ma\cos\phi}{l+m}}{\dfrac{(l+m)\cos\phi}{ma}} = \frac{y - \dfrac{lb\sin\phi}{l+m}}{\dfrac{(l+m)\sin\phi}{lb}},$$

i.e.

$$\frac{max}{\cos\phi} - \frac{lby}{\sin\phi} = \frac{m^2a^2 - l^2b^2}{l+m}.$$

If this is identical with

$$\frac{x}{a\cos\phi} + \frac{y}{b\sin\phi} = 1,$$

we have $\qquad ma^2 = -lb^2. \qquad \therefore \dfrac{l}{m} = -\dfrac{a^2}{b^2}.$

Ex. 11. Show that if the portion of the polar of (x', y') with respect to the ellipse $b^2x^2+a^2y^2 = a^2b^2$ which is intercepted by it subtends a right angle at the centre, the point (x', y') lies on the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4} = \frac{1}{a^2}+\frac{1}{b^2}$. [B.]

The lines joining the centre to the points where the line $\frac{xx'}{a^2}+\frac{yy'}{b^2} = 1$ meets the ellipse are

$$b^2x^2+a^2y^2 = \left(\frac{xx'}{a^2}+\frac{yy'}{b^2}\right)^2 a^2b^2.$$

If these are perpendicular, the sum of the coefficients of x^2 and y^2 is zero; hence

$$b^2\left(1-\frac{b^2x'^2}{a^2}\right)+a^2\left(1-\frac{a^2y'^2}{b^2}\right) = 0,$$

i.e.
$$\frac{x'^2}{a^4}+\frac{y'^2}{b^4} = \frac{1}{a^2}+\frac{1}{b^2}.$$

Ex. 12. *AA'* is the major axis of an ellipse and *P* is the point ϕ on it. *A'P*, *AP* meet the tangents at *A*, *A'* in *Q* and *R* respectively. Prove that the equation of *QR* is

$$\frac{x}{a}\cos\phi+\frac{y}{2b}\sin\phi = 1$$

and find its envelope. [O. and C.]

The equation of *AP* is

$$\frac{x}{a}\cos\frac{\phi}{2}+\frac{y}{b}\sin\frac{\phi}{2} = \cos\frac{\phi}{2},$$

i.e.
$$y = -\frac{b}{a}\cot\frac{\phi}{2}(x-a).$$

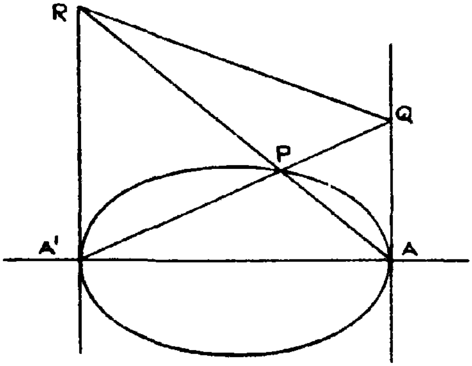
Where this cuts $x = -a$, we have $y = 2b\cot\frac{\phi}{2}$.

So the equation of *A'P* is $y = \frac{b}{a}\tan\frac{\phi}{2}(x+a)$.

Where this cuts $x = a$, $y = 2b\tan\frac{\phi}{2}$.

The equation of QR is now

$$\frac{x+a}{-2a} = \frac{y-2b\cot\frac{\phi}{2}}{2b\left(\cot\frac{\phi}{2}-\tan\frac{\phi}{2}\right)} = \frac{\frac{y}{2}\sin\phi - 2b\cos^2\frac{\phi}{2}}{2b\cos\phi},$$



which reduces to

$$b(x+a)\cos\phi + \frac{ay}{2}\sin\phi = ab(1+\cos\phi),$$

i.e.
$$\frac{x}{a}\cos\phi + \frac{y}{2b}\sin\phi = 1.$$

The form of this equation shows that it is a tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{4b^2} = 1,$$

which is therefore its envelope.

EXAMPLES 12

1. Find the ordinates of the points on the ellipse $9x^2+25y^2 = 225$ whose abscissae are 2, -3, and $-1\frac{1}{2}$ respectively.

2. Find the eccentricity of the ellipse whose axes are 16 and 12.

3. Find the distance between the foci of the ellipse $2x^2+3y^2=6$.

4. Find the distance between the directrices of the ellipse $9x^2+11y^2=99$.

5. The length of the major axis of an ellipse is 16 and its foci are the points $(\pm 4, 0)$; find the equation of the ellipse.

6. The foci of an ellipse are the points $\left(\pm\frac{2\sqrt{2}}{3}, 0\right)$ and its eccentricity is $\frac{1}{3}$. Find its equation.

7. The axes of an ellipse lie on the axes of coordinates and it passes through the points $(7\frac{1}{5}, 3)$, $(5\frac{2}{5}, -4)$. Find its equation.

8. Find the eccentricity, the equations of the directrices, and the lengths of a latus rectum of the following ellipses:

(1) $9x^2+25y^2-225=0$.
(2) $9(x-2)^2+36(y+3)^2=324$.
(3) $2x^2+3y^2-8x-18y+29=0$.

9. Find the equation of an ellipse whose minor axis is 8 and latus rectum 4.

10. Find the equation of the ellipse which has the point $(2, 3)$ for its focus, the line $5x-12y=0$ for its directrix, and whose eccentricity is $\frac{1}{2}$.

11. Find the coordinates of the foci, the length of the latus rectum, and the equations of the directrices of the ellipse whose equation is $4(x^2+y^2)=(x\cos 60^\circ+y\sin 60^\circ-1)^2$.

12. What is the eccentricity of an ellipse whose latus rectum is equal to half of its major axis?

13. If S is the focus of an ellipse whose axes are ACA', BCB', prove that $SB=SB'=CA$.

14. What are the eccentric angles of the extremities of the latera recta of the ellipse $2x^2+4y^2=1$?

15. The ordinate of any point P on an ellipse cuts the auxiliary circle in p. A line PLl is drawn parallel to Cp cutting the axes in L and l. Prove that $LP=CB$ and $lp=CA$.

16. Three points P, Q, R are taken on a straight line such that $PQ=3QR$. If P and Q move on two perpendicular straight lines, find the equation of the locus R traces out with regard to these lines as axes of coordinates.

17. A point moves so that the sum of its distances from the two points $(\pm l, 0)$ is equal to $2k$, where k is constant. Prove that the equation of its locus is

$$\frac{x^2}{k^2}+\frac{y^2}{k^2-l^2}=1.$$

18. An ellipse is drawn to pass through the points $(1, 12)$, $(1, -4)$, and $(8, 10)$, and to have the line $x = 4$ as an axis of symmetry. Find the coordinates of its foci.

19. Find the equation of the ellipse $9x^2+25y^2 = 225$ referred to its major axis and the tangent at its positive extremity as axes of coordinates.

20. An ellipse has one focus at the point $(2, 4)$ and its major axis parallel to the x-axis. It also touches both axes of coordinates. Find its equation.

21. If ACA' be the major axis of an ellipse and PN the ordinate to ACA' from any point P on the ellipse, show that

$$PN^2 : AN.A'N = b^2 : a^2.$$

22. PNP' is drawn through a point N on the major axis of an ellipse perpendicular to this axis and to cut the ellipse in P and P'. Any other point Q is taken on the ellipse and QP, QP' produced, if necessary, cut the major axis in R and R' respectively. Prove that $CR.CR' = a^2$.

23. Write down the equation of the chord of the ellipse $x^2+2y^2 = 1$ joining two points on it whose eccentric angles are 30° and 60°.

24. Find the equation of the tangent and normal to the following ellipses at the points stated:

(1) $9x^2+25y^2 = 225$, at the point $(-\frac{5}{3}, 2\sqrt{2})$.

(2) $\dfrac{x^2}{64}+\dfrac{y^2}{49} = 1$, at the point $(8\cos\alpha, 7\sin\alpha)$.

(3) $\dfrac{(x-3)^2}{36}+\dfrac{(y+4)^2}{9} = 1$, at the point $(5, 2\sqrt{2}-4)$.

25. If $y = \sqrt{5}x+c$ is a tangent to the ellipse $4x^2+9y^2 = 36$, what is c?

26. The lines whose equations are $y = 2x+\sqrt{65}$, $y = 3x+\sqrt{105}$, touch the ellipse $b^2x^2+a^2y^2 = a^2b^2$. Find a and b.

27. Find the equations of the tangents to the ellipse

$$4x^2+9y^2 = 36$$

which make equal angles with the major and minor axes.

28. Find the least value of the portion of a tangent to the ellipse $b^2x^2+a^2y^2 = a^2b^2$ intercepted between the axes.

29. Find the equation of the director circle of the ellipse $2x^2+y^2 = 5$.

30. Find the coordinates of the centres of the following ellipses:

(1) $(2x-3)^2+(3y-2)^2 = 8$.

(2) $5x^2+4y^2-20x+18y+32 = 0$.

(3) $7x^2-9xy+3y^2-6x+4y-2 = 0$.

31. P is a point on the ellipse $b^2x^2+a^2y^2 = a^2b^2$ whose eccentric angle is ϕ. The tangent and normal at P to the ellipse cut the major axis in T and G respectively. PN is drawn perpendicular to this axis. Find the lengths of the subtangent NT and of the subnormal NG.

32. With the usual notation (see figure of Art. 9) prove that

$$(1)\ \frac{1}{SY}+\frac{1}{S'Y'} = \frac{2}{PG}, \qquad (2)\ \frac{CG^2}{b^2}+\frac{Cg^2}{a^2} = \frac{a^2e^2}{b^2}.$$

33. If $x = h+a\cos t$ and $y = k+b\sin t$, where t varies, find the equation of the locus of (x, y).

34. An ellipse is drawn with its foci at the points $(\pm c, 0)$. Prove that, if a straight line $\frac{x}{p}+\frac{y}{q} = 1$ cuts the ellipse at two points whose eccentric angles are α and β, then

$$c^2\cos(\alpha-\beta)-(p^2-q^2)\cos(\alpha+\beta) = p^2-q^2-c^2. \qquad \text{[J.]}$$

35. Find the coordinates of the points of intersection of the ellipse $3x^2+4y^2 = 12$ with the circle $x^2+y^2 = 3\frac{11}{27}$. Find also the angles at which they intersect at their common points.

36. Any tangent to an ellipse cuts the tangents at the extremities of its major axis in L and M. If S be a focus, show that $\angle LSM$ is a right angle.

37. Prove that the tangents to an ellipse $b^2x^2+a^2y^2 = a^2b^2$ which are equally inclined to the axes form a square of area $2(a^2+b^2)$.

38. What is the condition that the line $\frac{x}{\alpha}+\frac{y}{\beta}=1$ should touch the ellipse $b^2x^2+a^2y^2=a^2b^2$?

39. At what points on an ellipse $b^2x^2+a^2y^2=a^2b^2$ are the tangents equally inclined to the axes?

40. Show that the line $y=mx-\frac{m(a^2-b^2)}{a^2+b^2m^2}$ is a normal to the ellipse $b^2x^2+a^2y^2=a^2b^2$, for any value of m.

41. Show that the point $\frac{a^2-b^2}{a^2+b^2}a\cos\phi,\ -\frac{a^2-b^2}{a^2+b^2}b\sin\phi$, lies on the normal at a point on the ellipse $b^2x^2+a^2y^2=a^2b^2$ whose eccentric angle is ϕ.

42. If the normals at the points whose eccentric angles are α and β meet on the line $y=x$, prove that

$$a\sin\alpha\sin\beta\sin\tfrac{1}{2}(\alpha+\beta)+b\cos\alpha\cos\beta\cos\tfrac{1}{2}(\alpha+\beta)=0.$$

43. A normal to an ellipse makes an angle of 45° with the major axis. Prove that the area of the triangle it forms with the axes is $\frac{(a^2-b^2)^2}{2(a^2+b^2)}$.

44. An ordinate NP produced cuts the tangent at an extremity of the latus rectum of an ellipse in Q. Prove that $SP=NQ$.

45. Prove that the perpendicular bisectors of the chords which join two fixed points on an ellipse to any variable third point intercept a constant length upon the major axis.

46. The abscissa of any point on an ellipse is $\frac{a}{e}\cos\theta$. Show that the perpendicular from the focus $(ae, 0)$ on the tangent at the point is of length $b\tan\frac{\theta}{2}$.

47. P is any point on an ellipse and p the corresponding point on the auxiliary circle. Find the coordinates of the point of intersection of the normals to the ellipse at P and to the auxiliary circle at p in terms of the eccentric angle ϕ of the point P.

48. Find the perpendicular distance p of a tangent at a point on the ellipse $b^2x^2+a^2y^2=a^2b^2$ from the centre in terms of θ, the angle the perpendicular makes with the major axis.

49. Show that if p and q are the lengths of perpendiculars from the centre of an ellipse on the tangent and normal at any point, then $p^2q^2 = (a^2-p^2)(p^2-b^2)$.

50. TP, TQ are the tangents from a point T to an ellipse, and SY is drawn perpendicular to PQ from the focus S. Show that SY and CT meet on the directrix.

51. P is any point on an ellipse and p the corresponding point on its auxiliary circle. Show that the length of the perpendicular from the focus S on the tangent at p to the circle is equal to SP.

52. Two sides of a triangle inscribed in an ellipse are drawn in fixed directions, show that the third side must always touch another ellipse.

53. Find the coordinates of the pole with regard to an ellipse $b^2x^2+a^2y^2 = a^2b^2$ of the following lines:

(1) $lx+my = 1$. (2) $x\cos\alpha+y\sin\alpha = p$.

(3) The line joining the positive vertices of the major and minor axes.

(4) The normal at a point whose eccentric angle is ϕ.

54. From a point T on the auxiliary circle tangents TP and TQ are drawn to the ellipse. If S and S' are the foci, prove that SP is parallel to $S'Q$.

55. From a point P on an ellipse PNP' is drawn perpendicular to the major axis and cutting the ellipse again in P'. The normal at P cuts CP' in Q. Find the locus of Q.

56. From the centre of an ellipse a perpendicular CK is drawn to a focal chord PSQ. The tangents at P and Q meet at a point T on the directrix. Prove that $TS.CK = b^2$.

57. The tangent at any point of an ellipse meets the tangents at the extremities of the major axis in P and Q. Show that the locus of the point of intersection of the lines PS and QS', where S and S' are the foci of the ellipse, is an ellipse.

58. A point P is taken on an ellipse whose foci are S and S'. PS and PS' are produced to cut the ellipse again in points θ and ϕ. Prove that $\tan\frac{\theta}{2} : \tan\frac{\phi}{2}$ is a constant ratio.

59. Parallel lines through the foci of an ellipse meet the tangent

at the vertex A in P and Q. $A'P$, $A'Q$ meet the auxiliary circle in H and K. Prove that HK touches the ellipse.

60. T is any point on the tangent at P to an ellipse from which perpendiculars TM, TN are drawn to SP and to the directrix. Show that $SM = eTN$.

61. SY is the perpendicular from the focus S of an ellipse on any tangent to it. K is the point on SY produced such that $SY = YK$. Prove that the square of the length of the tangent from K to the director circle $= 2SY^2$.

62. CY is the perpendicular from the centre of an ellipse to the tangent at the point on it whose eccentric angle is ϕ. A line is drawn through the focus S parallel to the tangent, cutting CY in K. Show that $\dfrac{CK}{CY} = e\cos\phi$.

63. P is any point on an ellipse whose major axis is ACA'. A line is drawn through the point $(-a, 2a)$ perpendicular to AP; find the locus of its intersection with a line through A' drawn perpendicular to $A'P$.

64. The polar of any point P on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 4$ with regard to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ cuts the latter ellipse in the points Q and R. Prove that the centroid of the triangle PQR lies on it also.

65. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$, $\dfrac{x^2}{a}+\dfrac{y^2}{b} = a+b$ are two concentric ellipses. Prove that the pole of any tangent to the first with regard to the second lies on the director circle of the latter.

66. The normal at any point P of an ellipse cuts the major axis in G. Find the locus of the middle point of PG.

67. The tangents at two points (x_1, y_1), (x_2, y_2) on the ellipse $b^2x^2+a^2y^2 = a^2b^2$ meet at the point (α, β), and the normals at the same points meet at the point (h, k). Show that $ha^2 = e^2\alpha x_1 x_2$ and $kb^4 = -a^2e^2\beta y_1 y_2$.

68. Prove that the locus of the middle points of normal chords of an ellipse is $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)^2\left(\dfrac{a^6}{x^2}+\dfrac{b^6}{y^2}\right) = a^4e^4$.

69. PSQ is a focal chord of an ellipse. The tangent at P and the normal at Q intersect in R. Prove that the minor axis bisects QR.

70. Assuming that the centroid of an equilateral triangle inscribed in an ellipse is the centre of the ellipse, show that the eccentric angles of its vertices are α, $\alpha+\frac{2\pi}{3}$, $\alpha+\frac{4\pi}{3}$; if α be the eccentric angle of one of them.

71. If the eccentric angles of three points on an ellipse are α, β, and γ, show that the area of the triangle formed by joining these points is

$$2ab\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}.$$

72. Find the polar equation of the locus of the foot of the perpendicular drawn from the centre of an ellipse to the tangent at any point on it. Deduce its cartesian equation from this.

73. Find the polar equations of the directrices of the ellipse whose equation is $\frac{l}{r}=1-e\cos\theta$.

74. The difference of the angles two focal radii make with the major axis of an ellipse is 2λ. Find the polar equation of the locus of the point where the tangents at their extremities meet.

75. Find the polar equation of a chord of an ellipse which subtends a right angle at the focus.

76. A line is drawn parallel to the minor axis of an ellipse halfway between S and its directrix. If PSQ is any focal chord, show that the perpendiculars on the line from P and Q have a constant product.

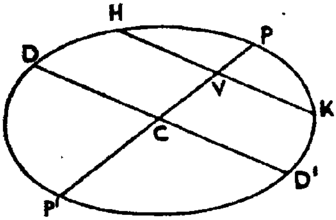
77. If the tangent at any point of an ellipse makes an angle α with its major axis and one of β with the focal radius of the point of contact, show that $e\cos\alpha=\cos\beta$.

78. AA' is the major axis of an ellipse whose focus is S. A parabola is described with S as focus and passing through the extremities of the minor axis of the ellipse. Prove that the vertex of this parabola bisects either SA or SA'.

79. The polar of a point T with regard to an ellipse cuts the directrix corresponding to the focus S in R; prove that $\angle TSR$ is a right angle.

30. A circle of fixed radius passing through the focus of an ellipse cuts it in four points whose focal radii are r_1, r_2, r_3, r_4. Prove that $r_1 r_2 r_3 r_4$ is constant.

13. Conjugate diameters of an ellipse



We shall prove first of all that the locus of the middle points of a system of parallel chords of an ellipse is a diameter of the ellipse.

Let H, K be the points ϕ, ϕ' on an ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Then the equation of the chord HK is

$$\frac{x}{a}\cos\frac{\phi+\phi'}{2}+\frac{y}{b}\sin\frac{\phi+\phi'}{2}=\cos\frac{\phi-\phi'}{2}.$$

If HK is drawn in a specified direction, i.e. if its gradient m is constant, we have

$$m=-\frac{b}{a}\cot\frac{\phi+\phi'}{2}.$$

Suppose that the coordinates of V, the middle point of HK, are X and Y. Then

$$X=\tfrac{1}{2}a(\cos\phi+\cos\phi')=a\cos\frac{\phi+\phi'}{2}\cos\frac{\phi-\phi'}{2},$$

$$Y=\tfrac{1}{2}b(\sin\phi+\sin\phi')=b\sin\frac{\phi+\phi'}{2}\cos\frac{\phi-\phi'}{2}.$$

$$\therefore\ \frac{Y}{X}=\frac{b}{a}\tan\frac{\phi+\phi'}{2}=-\frac{b^2}{a^2m}.$$

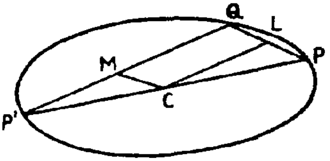
So that the locus of V is the diameter $y=m'x$, where

$$\mathbf{mm'=-\frac{b^2}{a^2}}.$$

The diameter of an ellipse which bisects all chords parallel to another diameter is said to be *conjugate* with regard to that diameter.

If $y = mx$ and $y = m'x$ are conjugate diameters, then, as just proved, $mm' = -\frac{b^2}{a^2}$. The fact that m and m' can be interchanged in this relation without affecting it, shows that, if $y = mx$ is conjugate with regard to $y = m'x$, then $y = m'x$ will be conjugate with regard to $y = mx$.

This is the same as asserting that, if a diameter bisects all chords parallel to another diameter, then the second diameter will bisect all chords parallel to the first diameter.



If PCP' be any diameter of an ellipse and Q be any point on it, then QP, QP' are called *Supplemental Chords.*

Supplemental chords such as QP, QP' are always parallel to a pair of conjugate diameters.

Bisect QP, QP' in L and M respectively and join CL, CM. Then, L being the middle point of PQ, CL bisects all chords parallel to PQ. And as CM is parallel to PQ, $\therefore$ CL bisects all chords parallel to CM. $\therefore$ CL, CM are conjugate diameters.

$\therefore$ QP, QP', which are parallel to CM, CL respectively, are parallel to a pair of conjugate diameters.

14. Coordinates of the extremities of two conjugate diameters

Let CP, CD be two conjugate semidiameters of an ellipse and let the eccentric angles of P and D be ϕ and ϕ'. Then $\phi' \sim \phi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

The equation of CP is $\frac{x}{a\cos\phi} = \frac{y}{b\sin\phi}$, i.e. $y = \frac{b}{a}\tan\phi$.

That of CD is, similarly, $y = \frac{b}{a}\tan\phi'$.

As CP, CD are conjugate, we have

$$\frac{b^2}{a^2}\tan\phi\tan\phi' = -\frac{b^2}{a^2}.$$

$$\therefore\ 1+\tan\phi\tan\phi' = 0. \qquad \therefore\ \tan(\phi\sim\phi') = \infty.$$

$$\therefore\ \phi'\sim\phi = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}.$$

Suppose that CP is in the first quadrant and CD in the second, then the coordinates of P being $a\cos\phi$, $b\sin\phi$, those of D are $a\cos\left(\phi+\frac{\pi}{2}\right)$, $b\sin\left(\phi+\frac{\pi}{2}\right)$, i.e. $-a\sin\phi$, $b\cos\phi$.

15. Some properties of conjugate diameters

(1). *The tangent at P, the extremity of a diameter PCP' of an ellipse, is parallel to the diameter DCD' conjugate to PCP'.*

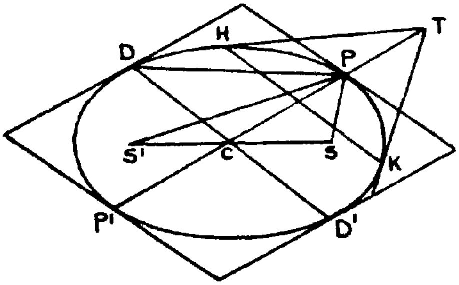
The equation of the tangent at P, the point ϕ, is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi = 1,$$

and that of CD, the line joining $(0,0)$, $(-a\sin\phi, b\cos\phi)$, is

$$\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b} = 0,$$

which two lines are clearly parallel.



Hence the tangents at the extremities of a pair of conjugate diameters form a parallelogram circumscribing the ellipse.

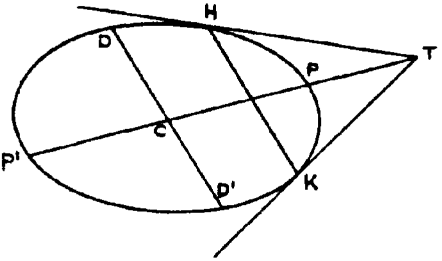
(2) *If HK is a chord of an ellipse parallel to a diameter DCD', the tangents at H and K intersect on the diameter PCP' conjugate to DCD'.*

Let H and K be the points θ, θ' on the ellipse.

The equation of HK is $\frac{x}{a}\cos\frac{\theta+\theta'}{2}+\frac{y}{b}\sin\frac{\theta+\theta'}{2}=\cos\frac{\theta-\theta'}{2}$, and, as it is parallel to DCD', viz. to the line

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=0,$$

$$\therefore\ \cot\frac{\theta+\theta'}{2}=\cot\phi. \qquad \therefore\ \frac{\theta+\theta'}{2}=\phi.$$



Now the coordinates of the point where the tangents at θ and θ' intersect are

$$\frac{a\cos\frac{\theta+\theta'}{2}}{\cos\frac{\theta-\theta'}{2}},\quad \frac{b\sin\frac{\theta+\theta'}{2}}{\cos\frac{\theta-\theta'}{2}},$$

i.e.
$$\frac{a\cos\phi}{\cos\frac{\theta-\theta'}{2}},\quad \frac{b\sin\phi}{\cos\frac{\theta-\theta'}{2}},$$

which clearly satisfy the equation of PCP', viz.

$$\frac{x}{a\cos\phi}=\frac{y}{b\sin\phi}.$$

Hence the pole of HK lies on CP if HK is parallel to the conjugate diameter DCD'.

(3) If CP, CD are two conjugate semidiameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $CP^2+CD^2=a^2+b^2$.

The coordinates of P and D being

$$(a\cos\phi, b\sin\phi),\ (-a\sin\phi, b\cos\phi)$$

respectively,
$$CP^2 = a^2\cos^2\phi + b^2\sin^2\phi,$$
$$CD^2 = a^2\sin^2\phi + b^2\cos^2\phi.$$

Adding: $\mathbf{CP^2 + CD^2 = a^2 + b^2}$.

(4) The area of the triangle $CPD = \frac{1}{2}ab$.

Taking P and D as before,

$$PD^2 = a^2(\cos\phi + \sin\phi)^2 + b^2(\sin\phi - \cos\phi)^2.$$

The equation of PD, i.e. of the line joining the points whose eccentric angles are ϕ and $\phi + \frac{\pi}{2}$, is

$$\frac{x}{a}\cos\frac{\phi+\phi+\frac{\pi}{2}}{2} + \frac{y}{b}\sin\frac{\phi+\phi+\frac{\pi}{2}}{2} = \cos\frac{\phi-\phi-\frac{\pi}{2}}{2},$$

i.e.
$$\frac{x}{a}\cos\left(\frac{\pi}{4}+\phi\right) + \frac{y}{b}\sin\left(\frac{\pi}{4}+\phi\right) = \cos\frac{\pi}{4},$$

i.e.
$$\frac{x}{a}(\cos\phi - \sin\phi) + \frac{y}{b}(\cos\phi + \sin\phi) = 1.$$

The perpendicular from the centre C $(0,0)$ on this line is

$$\frac{1}{\sqrt{\left\{\frac{(\cos\phi - \sin\phi)^2}{a^2} + \frac{(\cos\phi + \sin\phi)^2}{b^2}\right\}}} = \frac{ab}{PD}.$$

But the area of triangle CPD is half this perpendicular × PD

$$= \frac{1}{2}\frac{ab}{PD}\cdot PD = \tfrac{1}{2}ab.$$

The area of the circumscribing parallelogram, formed by the tangents at the extremities of two conjugate diameters PCP', DCD', is clearly 8 times that of the triangle CPD and is therefore 4ab.

(5) To prove that $SP.S'P = CD^2$, where S and S' are the foci of the ellipse and CP and CD conjugate semidiameters.

$$SP = e\left(\frac{a}{e} - a\cos\phi\right) = a(1 - e\cos\phi) \quad \text{[Art. 9 (4)]}$$

and $$S'P = e\left(\frac{a}{e} + a\cos\phi\right) = a(1 + e\cos\phi).$$

$$\therefore\ SP.S'P = a^2(1 - e^2\cos^2\phi) = a^2 - (a^2 - b^2)\cos^2\phi$$
$$= a^2\sin^2\phi + b^2\cos^2\phi$$
$$= CD^2.$$

(6) When CP and CD are conjugate semidiameters and $CP = CD$, then CP and CD are called the *equi-conjugate diameters.*

With the previous notation

$$CP^2 = a^2\cos^2\phi + b^2\sin^2\phi$$

and $$CD^2 = a^2\sin^2\phi + b^2\cos^2\phi.$$

If $CP^2 = CD^2$, we have $(a^2 - b^2)(\cos^2\phi - \sin^2\phi) = 0$.

$$\therefore\ \tan^2\phi = 1 \quad \text{and} \quad \phi = \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4}.$$

Also CP^2 and CD^2 both have the value $\dfrac{a^2 + b^2}{2}$.

The equations of the equi-conjugate diameters are

$$y = \pm\frac{b}{a}\tan\frac{\pi}{4}x, \quad \text{i.e. } y = \pm\frac{b}{a}x,$$

or as one equation $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

16. Conormal points of an ellipse

The equation of the normal at a point ϕ on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2.$$

If this normal passes through the point (h, k), we have

$$\frac{ah}{\cos\phi}-\frac{bk}{\sin\phi}=a^2-b^2.$$

In this equation put $\tan\frac{\phi}{2}=t$, and it becomes

$$\frac{ah(1+t^2)}{1-t^2}-\frac{bk(1+t^2)}{2t}=a^2-b^2,$$

which reduces to

$$bkt^4+2(ah+a^2-b^2)t^3+2(ah-a^2+b^2)t-bk=0.$$

This is an equation of the fourth degree in t, and will have four roots t_1, t_2, t_3, t_4 suppose, which give the points on the ellipse whose normals pass through the point (h, k).

Let s_r be the symbol for the sum of all the products of n quantities $t_1, t_2,..., t_n$ taken r at a time. Then, assuming the algebraical relations between the roots of an equation and its coefficients, we have in the present case

$$s_2=0,\quad \text{also } s_4=-\frac{bk}{bk}=-1. \qquad \therefore\ 1-s_2+s_4=0.$$

But $$\tan\frac{\phi_1+\phi_2+\phi_3+\phi_4}{2}=\frac{s_1-s_3}{1-s_2+s_4}=\infty.$$

$$\therefore\ \frac{\phi_1+\phi_2+\phi_3+\phi_4}{2}=n\pi+\frac{\pi}{2}$$

and $$\phi_1+\phi_2+\phi_3+\phi_4=(2n+1)\pi.$$

Hence the sum of the eccentric angles of four points of an ellipse, the normals at which are concurrent, is always an odd multiple of two right angles. Any four points on an ellipse at which the normals are thus concurrent are known as *conormal points*.

The four examples following will illustrate the methods used in dealing with problems connected with conormal points on an ellipse.

(1) **To find the coordinates of the mean centre of the four points on an ellipse whose normals meet at the point (h, k).**

Taking the equation

$$\frac{ah}{\cos\phi}-\frac{bk}{\sin\phi}=a^2-b^2,$$

we have $$\left[\frac{ah}{\cos\phi}-(a^2-b^2)\right]^2=\frac{b^2k^2}{\sin^2\phi}=\frac{b^2k^2}{1-\cos^2\phi};$$

whence

$$(a^2-b^2)^2\cos^4\phi-2ah(a^2-b^2)\cos^3\phi+\text{other terms}=0.$$

If ϕ_1, ϕ_2, ϕ_3, ϕ_4 are the roots of this equation,

$$\cos\phi_1+\cos\phi_2+\cos\phi_3+\cos\phi_4=\frac{2ah}{a^2-b^2}.$$

And as $$\left[\frac{bk}{\sin\phi}+a^2-b^2\right]^2=\frac{a^2h^2}{1-\sin^2\phi},$$

$$\therefore\ (a^2-b^2)^2\sin^4\phi+2bk(a^2-b^2)\sin^3\phi+\text{other terms}=0;$$

hence $$\sin\phi_1+\sin\phi_2+\sin\phi_3+\sin\phi_4=-\frac{2bk}{a^2-b^2}.$$

Hence the coordinates of the mean centre of the four points ϕ_1, ϕ_2, ϕ_3, ϕ_4, which are

$$a\frac{\cos\phi_1+\cos\phi_2+\cos\phi_3+\cos\phi_4}{4},$$

$$b\frac{\sin\phi_1+\sin\phi_2+\sin\phi_3+\sin\phi_4}{4},$$

are seen to be $$\frac{a^2h}{2(a^2-b^2)}\quad\text{and}\quad-\frac{b^2k}{2(a^2-b^2)}.$$

(2) **If three normals at the points θ_1, θ_2, θ_3 on an ellipse meet at a point, then**

$$\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0.$$

The four normals which meet in (h, k) are given by the equation

$$bkt^4+2\{ah+(a^2-b^2)\}t^3+2(ah-a^2+b^2)t-bk = 0.$$

There is no coefficient of t^2 in this equation.

$$\therefore\ s_2 = t_1t_2+t_2t_3+t_3t_1+t_4(t_1+t_2+t_3) = 0.$$

Also $t_1t_2t_3t_4 = -1$.

$$\therefore\ t_1t_2+t_2t_3+t_3t_1 = \frac{1}{t_2t_3}+\frac{1}{t_1t_3}+\frac{1}{t_1t_2}.$$

Now

$$\frac{1}{t_1t_2}-t_1t_2 = \frac{1-t_1^2t_2^2}{t_1t_2} = \frac{(1-t_1t_2)(1+t_1t_2)}{t_1t_2}$$

$$= \frac{\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}-\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}+\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)}{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}$$

$$= \frac{\cos\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}}{\frac{1}{4}\sin\theta_1\sin\theta_2} = \frac{2(\cos\theta_1+\cos\theta_2)}{\sin\theta_1\sin\theta_2}.$$

Similarly, $$\frac{1}{t_2t_3}-t_2t_3 = \frac{2(\cos\theta_2+\cos\theta_3)}{\sin\theta_2\sin\theta_3}$$

and $$\frac{1}{t_1t_3}-t_1t_3 = \frac{2(\cos\theta_1+\cos\theta_3)}{\sin\theta_1\sin\theta_3}.$$

$$\therefore\ \frac{\cos\theta_1+\cos\theta_2}{\sin\theta_1\sin\theta_2}+\frac{\cos\theta_2+\cos\theta_3}{\sin\theta_2\sin\theta_3}+\frac{\cos\theta_1+\cos\theta_3}{\sin\theta_1\sin\theta_3} = 0;$$

whence $\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1) = 0.$

(3) **If the normals at three points of an ellipse meet at a point on a fixed normal to it, find the locus of the centroid of the triangle formed by the three points.**

Let ϕ_1, ϕ_2, ϕ_3 be the variable points and α the foot of the fixed normal.

Let (X, Y) be the centroid of the triangle whose vertices are the points ϕ_1, ϕ_2, ϕ_3.

Then $$3X = a(\cos\phi_1+\cos\phi_2+\cos\phi_3)$$

and $$3Y = b(\sin\phi_1+\sin\phi_2+\sin\phi_3).$$

But, as in the last example but one, if (h, k) is the point where the normals meet,

$$\cos\phi_1+\cos\phi_2+\cos\phi_3+\cos\alpha = \frac{2ah}{a^2-b^2}$$

and $$\sin\phi_1+\sin\phi_2+\sin\phi_3+\sin\alpha = -\frac{2bk}{a^2-b^2}.$$

$$\therefore\ ha = \frac{a^2-b^2}{2}\left(\frac{3X}{a}+\cos\alpha\right) \quad\text{and}\quad kb = -\frac{a^2-b^2}{2}\left(\frac{3Y}{b}+\sin\alpha\right).$$

But as (h, k) is on the normal at the point α,

$$\therefore\ \frac{ha}{\cos\alpha}-\frac{kb}{\sin\alpha} = a^2-b^2.$$

$$\therefore\ \frac{1}{2\cos\alpha}\left(\frac{3X}{a}+\cos\alpha\right)+\frac{1}{2\sin\alpha}\left(\frac{3Y}{b}+\sin\alpha\right) = 1.$$

$$\therefore\ Y+\frac{b}{a}X\tan\alpha = 0,$$

which is the equation of a fixed straight line through the centre of the ellipse.

(4) The normals at four points H, K, L, M on an ellipse are concurrent. Show that the inscribed parallelogram, which has its diagonals parallel to HK and LM, has its sides parallel to the equi-conjugate diameters.

Suppose that H, K, L, M are four points t_1 t_2, t_3, t_4 on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $t_1=\tan\frac{\phi_1}{2}$, etc., ϕ_1, ϕ_2, ϕ_3, ϕ_4 being the eccentric angles of the four points.

The equation of the chord *HK* is

$$\frac{x}{a}(1-t_1 t_2)+\frac{y}{b}(t_1+t_2) = 1+t_1 t_2,$$

and that of the diameter parallel to it

$$\frac{x}{a} = -\frac{y}{b}\tan\frac{\phi_1+\phi_2}{2}.$$

So the equation of the diameter parallel to *LM* is

$$\frac{x}{a} = -\frac{y}{b}\tan\frac{\phi_3+\phi_4}{2}.$$

But $$\frac{\phi_1+\phi_2+\phi_3+\phi_4}{2} = n\pi+\frac{\pi}{2}.$$

∴ the diameters are

$$\frac{x}{a} = -\frac{y}{b}\tan\frac{\phi_1+\phi_2}{2}$$

and $$\frac{x}{a} = -\frac{y}{b}\cot\frac{\phi_1+\phi_2}{2}.$$

The vertices of the parallelogram are therefore the points

$$\pi-\frac{\phi_1+\phi_2}{2},\quad \frac{\pi}{2}+\frac{\phi_1+\phi_2}{2},\quad 2\pi-\frac{\phi_1+\phi_2}{2},\quad \frac{3\pi}{2}+\frac{\phi_1+\phi_2}{2}.$$

The chord joining the first two of these points is

$$\frac{x}{a}\cos\frac{3\pi}{4}+\frac{y}{b}\sin\frac{3\pi}{4} = \cos\left(\frac{\pi}{2}-\phi_1-\phi_2\right).$$

That joining the second and third

$$\frac{x}{a}\cos\frac{5\pi}{4}+\frac{y}{b}\sin\frac{5\pi}{4} = \cos\left(\frac{3\pi}{2}-\phi_1-\phi_2\right),$$

which are clearly parallel to $\frac{x}{a} = \pm\frac{y}{b}$, the equi-conjugate diameters.

Similarly for the lines joining the fourth point to the first and third.

17. Concyclic points on an ellipse

Take any circle whose equation is $x^2+y^2+2gx+2fy+c = 0$. The points where it cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ are given by putting $x = a\cos\phi$, $y = b\sin\phi$ in the equation of the circle. Making this substitution, we get

$$a^2\cos^2\phi+b^2\sin^2\phi+2ga\cos\phi+2fb\sin\phi+c = 0.$$

Put $\tan\frac{\phi}{2} = t$ in this equation and it becomes

$$a^2(1-t^2)^2+4b^2t^2+2ga(1-t^4)+4fbt(1+t^2)+c(1+t^2)^2 = 0,$$

which is an equation of the fourth degree in t whose four roots t_1, t_2, t_3, t_4 correspond to the four points in which the circle cuts the ellipse.

We observe that the coefficients of t^3 and of t in this equation are alike.

$$\therefore\ t_1+t_2+t_3+t_4 = t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4,$$

which relation can be written

$$\frac{t_1+t_2}{1-t_1t_2}+\frac{t_3+t_4}{1-t_3t_4} = 0,$$

i.e.
$$\tan\frac{\phi_1+\phi_2}{2}+\tan\frac{\phi_3+\phi_4}{2} = 0.$$

$$\therefore\ \frac{\phi_1+\phi_2}{2} = n\pi-\frac{\phi_3+\phi_4}{2},$$

i.e.
$$\phi_1+\phi_2+\phi_3+\phi_4 = 2n\pi.$$

The gradient of the chord joining any two of the points such as ϕ_1, ϕ_3 is $-\frac{b}{a}\cot\frac{\phi_1+\phi_3}{2}$, and that of the chord joining the remaining pair is $-\frac{b}{a}\cot\frac{\phi_2+\phi_4}{2}$. These being equal but of opposite sign, it follows that the chords are equally inclined to the major axis of the ellipse.

If two points of intersection between the circle and the ellipse become coincident, i.e. if the circle touches the ellipse, it follows from the foregoing that the tangent to either curve at the point of contact makes the same angle with the major axis as the chord joining the other two points of intersection does.

18. Equation of two tangents from a point

This is obtained in exactly the same way as in the case of the circle and the parabola.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let (h,k) be the point from which tangents are drawn.

The coordinates of the point dividing the line joining points (X,Y) and (h,k) in the ratio $p:q$ are

$$\frac{pX+qh}{p+q},\ \frac{pY+qk}{p+q}.$$

Substituting these coordinates in the equation of the ellipse we have

$$\frac{(pX+qh)^2}{a^2(p+q)^2}+\frac{(pY+qk)^2}{b^2(p+q)^2}=1,$$

which can be written

$$\frac{p^2}{q^2}\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}-1\right)+\frac{2p}{q}\left(\frac{Xh}{a^2}+\frac{Yk}{b^2}-1\right)+\frac{h^2}{a^2}+\frac{k^2}{b^2}-1=0.$$

If the line joining (X,Y) and (h,k) is a tangent to the ellipse, the roots of this equation in $\frac{p}{q}$ will be equal, the condition for which is

$$\left(\frac{Xh}{a^2}+\frac{Yk}{b^2}-1\right)^2=\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right),$$

which is the locus of (X,Y), any point on a tangent to the ellipse from (h,k).

The condition for the tangents to be perpendicular is that the sum of the coefficients of the X^2 and Y^2 terms in the foregoing equation should be zero. In this case

$$\frac{1}{a^2}\left(\frac{k^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{h^2}{a^2}-1\right)=0,$$

i.e. $$h^2+k^2=a^2+b^2.$$

So that in this case the point (h, k) would always lie on a circle concentric with the ellipse and of radius $\sqrt{(a^2+b^2)}$. The above, therefore, is another method of finding the equation of the Director Circle.

19. Envelopes

The method of finding envelopes, in the case of an ellipse, will be seen from the following four illustrations of it.

(1) Find the envelope of the polar of any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with regard to the parabola $y^2=4cx$.

The polar of any point $(a\cos\phi, b\sin\phi)$ with regard to the parabola is $yb\sin\phi=2c(x+a\cos\phi)$.

Put $\tan\frac{\phi}{2}=t$ and we have

$$\frac{yb.2t}{1+t^2}=2c\left[x+a\frac{1-t^2}{1+t^2}\right],$$

which can be rewritten

$$ct^2(x-a)-byt+c(x+a)=0.$$

The condition for equal roots then gives the equation of the envelope $$b^2y^2=4c^2(x^2-a^2).$$

(2) The polar of a point (h, k) with regard to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2+y^2=c^2$. Prove that (h, k) must be on the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$.

Any point on the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$ may be taken as $\frac{a^2}{c}\cos\phi, \frac{b^2}{c}\sin\phi$.

The polar of this point with regard to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{x\cos\phi}{c}+\frac{y\sin\phi}{c}=1,$$

which is known to be a tangent to the circle $x^2+y^2=c^2$, or could be proved to be so if necessary by putting $\tan\frac{\phi}{2}=t$ and expressing the condition for the resulting equation in t to have equal roots.

(3) The rectangle contained by the perpendiculars let fall on a straight line from its pole with respect to an ellipse and from the centre is constant. *Find the envelope of the line.*

Let the equation of the line be $\frac{xh}{a^2}+\frac{yk}{b^2}=1$.

The perpendicular from the pole is $\dfrac{\frac{h^2}{a^2}+\frac{k^2}{b^2}-1}{\sqrt{\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)}}$.

The perpendicular from the centre is $\dfrac{-1}{\sqrt{\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)}}$.

Suppose the product of these $=\lambda^2$, a constant, then

$$\frac{h^2}{a^2}\left(1+\frac{\lambda^2}{a^2}\right)+\frac{k^2}{b^2}\left(1+\frac{\lambda^2}{b^2}\right)=1.$$

Hence, θ being some angle, we can say

$$\cos^2\theta=\frac{h^2}{a^2}\left(1+\frac{\lambda^2}{a^2}\right) \quad\text{and}\quad \sin^2\theta=\frac{k^2}{b^2}\left(1+\frac{\lambda^2}{b^2}\right).$$

This being so, the equation of the line takes the form

$$\frac{x\cos\theta}{\sqrt{(a^2+\lambda^2)}}+\frac{y\sin\theta}{\sqrt{(b^2+\lambda^2)}}=1,$$

which always touches the ellipse

$$\frac{x^2}{a^2+\lambda^2}+\frac{y^2}{b^2+\lambda^2}=1.$$

(4) To find the envelope of a chord of an ellipse which subtends a constant angle, 2λ say, at the focus.

The equation of the chord joining the points whose vectorial angles are α and β on the ellipse $\frac{l}{r}=1-e\cos\theta$ is (Art. 12, Ex. 6)

$$\frac{l}{r}\cos\frac{\alpha-\beta}{2}=\cos\left(\theta-\frac{\alpha+\beta}{2}\right)-e\cos\frac{\alpha-\beta}{2}\cos\theta,$$

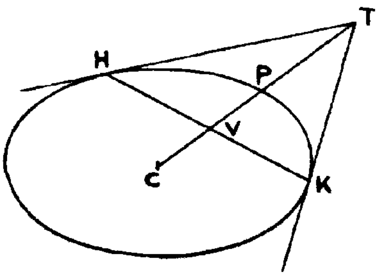
i.e. $$\frac{l}{r}\cos\lambda=\cos\left(\theta-\frac{\alpha+\beta}{2}\right)-e\cos\lambda\cos\theta,$$

which is the equation of a tangent to the ellipse of semi-latus rectum $l\cos\lambda$ and eccentricity $e\cos\lambda$, and which has the foci of the ellipse as its foci.

20. Typical examples

Ex. 1. TH, TK are two tangents to an ellipse, and CT cuts the chord HK in V and the curve in P. Prove that

$$CV.CT=CP^2.$$



It has been shown in Art. 15 (2) that CT is the semidiameter conjugate to the semidiameter parallel to HK. $\therefore$ CT bisects HK at V.

If H, K be the points ϕ_1, ϕ_2, the coordinates of V are $\frac{1}{2}a(\cos\phi_1+\cos\phi_2)$, $\frac{1}{2}b(\sin\phi_1+\sin\phi_2)$.

These are equal to

$$a\cos\frac{\phi_1+\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2} \quad\text{and}\quad b\sin\frac{\phi_1+\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2}.$$

$$\therefore\ CV^2=\cos^2\frac{\phi_1-\phi_2}{2}\left(a^2\cos^2\frac{\phi_1+\phi_2}{2}+b^2\sin^2\frac{\phi_1+\phi_2}{2}\right).$$

The coordinates of T are $\dfrac{a\cos\frac{\phi_1+\phi_2}{2}}{\cos\frac{\phi_1-\phi_2}{2}}$, $\dfrac{b\sin\frac{\phi_1+\phi_2}{2}}{\cos\frac{\phi_1-\phi_2}{2}}$.

$$\therefore\ CT^2=\frac{a^2\cos^2\frac{\phi_1+\phi_2}{2}+b^2\sin^2\frac{\phi_1+\phi_2}{2}}{\cos^2\frac{\phi_1-\phi_2}{2}}.$$

$$\therefore\ CV.CT=a^2\cos^2\frac{\phi_1+\phi_2}{2}+b^2\sin^2\frac{\phi_1+\phi_2}{2}.$$

The tangent at P is parallel to HK and its equation is

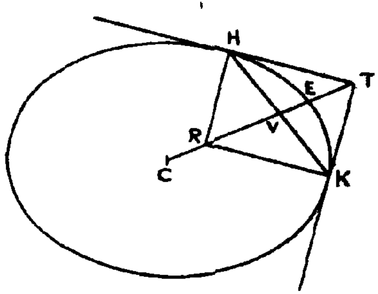
$$\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}+\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=1.$$

P is the point $a\cos\frac{\phi_1+\phi_2}{2}$, $b\sin\frac{\phi_1+\phi_2}{2}$.

$$\therefore\ CP^2=a^2\cos^2\frac{\phi_1+\phi_2}{2}+b^2\sin^2\frac{\phi_1+\phi_2}{2}=CV.CT.$$

Ex. 2. To find the locus of the point of intersection of two perpendicular normals of an ellipse.

If the normals at two points are perpendicular, so will be the tangents. Let the points be H, K and let the tangents meet at T. If the normals at H, K meet at R, R will clearly lie on CT and $THRK$ will be a rectangle whose diagonals intersect in V. Let CT cut the curve in E.



In the diagram

$$CR = CV - VR = CV - VT = 2CV - CT$$

$$= \frac{2CE^2}{CT} - CT,$$

since $$CV.CT = CE^2.$$

T being on the director circle, $CT^2 = a^2+b^2$.

If r be the length of CE and CE makes an angle of θ with the major axis, then

$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1.$$

$$\therefore\ r^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}.$$

$$\therefore\ CR = 2\frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}\frac{1}{\sqrt{(a^2+b^2)}} - \sqrt{(a^2+b^2)},$$

i.e. $$r = \frac{2a^2b^2 - (a^2\sin^2\theta + b^2\cos^2\theta)(a^2+b^2)}{\sqrt{(a^2+b^2)(a^2\sin^2\theta + b^2\cos^2\theta)}}$$

is the polar equation of the locus of R.

It reduces to

$$r = \frac{(b^2\cos^2\theta - a^2\sin^2\theta)(a^2-b^2)}{\sqrt{(a^2+b^2)(a^2\sin^2\theta + b^2\cos^2\theta)}}.$$

Squaring and putting

$$x^2+y^2 = r^2,\quad x^2 = r^2\cos^2\theta,\quad y^2 = r^2\sin^2\theta,$$

we have

$$(x^2+y^2)(a^2+b^2)(a^2y^2+b^2x^2)^2 = (a^2-b^2)^2(b^2x^2-a^2y^2)^2,$$

i.e.
$$\frac{(x^2+y^2)(a^2+b^2)}{(a^2-b^2)^2} = \left(\frac{b^2x^2-a^2y^2}{b^2x^2+a^2y^2}\right)^2,$$

which is the equation of the required locus.

Ex. 3. Show that if a circle cut an ellipse in four points, and three of them are conormal with another point on the ellipse, then this point and the fourth point of intersection of the circle and the ellipse will be at opposite ends of a diameter of the ellipse.

If ϕ_1, ϕ_2, ϕ_3, ϕ_4 be the four points of intersection of the circle and the ellipse, we have, as in Art. 17,

$$\phi_1+\phi_2+\phi_3+\phi_4 = 2n\pi,$$

and if ϕ_1, ϕ_2, ϕ_3, θ are points at which the normals are concurrent, we have, as in Art. 16,

$$\phi_1+\phi_2+\phi_3+\theta = (2n+1)\pi.$$

$$\therefore\ \theta-\phi_4 \text{ is an odd multiple of } \pi,$$

i.e. θ and ϕ_4 are at opposite extremities of a diameter of the ellipse.

Ex. 4. If $lx+my = 1$ is a chord of an ellipse on which a circle is described as diameter, prove that the equation of the other chord common to the circle and the ellipse is

$$lx-my = \frac{a^2+b^2}{a^2-b^2}.$$

Let ϕ_1, ϕ_2, ϕ_3, ϕ_4 be the four points of intersection of the circle and the ellipse. Then the equations of the chords are

$$\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}+\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2} = \cos\frac{\phi_1-\phi_2}{2}$$

and
$$\frac{x}{a}\cos\frac{\phi_3+\phi_4}{2}+\frac{y}{b}\sin\frac{\phi_3+\phi_4}{2} = \cos\frac{\phi_3-\phi_4}{2}.$$

They are equally inclined to the axis (Art. 17).

∴ the second equation becomes

$$\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}-\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=-\cos\frac{\phi_3-\phi_4}{2}.$$

The chords joining ϕ_3 and ϕ_1, ϕ_3 and ϕ_2 are perpendicular.

$$\therefore\quad \frac{\cos\frac{\phi_3+\phi_1}{2}\cos\frac{\phi_3+\phi_2}{2}}{\sin\frac{\phi_3+\phi_1}{2}\sin\frac{\phi_3+\phi_2}{2}}=-\frac{b^2}{a^2}.$$

$$\therefore\quad \frac{a^2-b^2}{a^2+b^2}=-\frac{\cos\frac{\phi_1-\phi_2}{2}}{\cos\frac{\phi_1+\phi_2+2\phi_3}{2}}=-\frac{\cos\frac{\phi_1-\phi_2}{2}}{\cos\frac{\phi_3-\phi_4}{2}}.$$

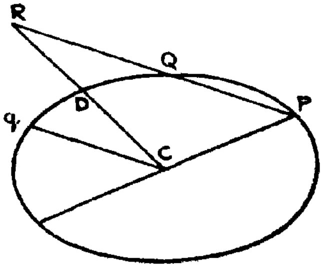
∴ the equation of the second chord can be written

$$\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}-\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=\frac{a^2+b^2}{a^2-b^2}\cos\frac{\phi_1-\phi_2}{2}.$$

But $$l=\frac{\cos\frac{\phi_1+\phi_2}{2}}{a\cos\frac{\phi_1-\phi_2}{2}}\quad\text{and}\quad m=\frac{\sin\frac{\phi_1+\phi_2}{2}}{b\cos\frac{\phi_1-\phi_2}{2}}.$$

$$\therefore\ lx-my=\frac{a^2+b^2}{a^2-b^2}.$$

Ex. 5. If any chord PQ of an ellipse cut the diameter conjugate to CP in R and Cq be the semi-diameter parallel to the chord, then $PQ.PR=2Cq^2$.



Let P and Q be the points θ and ϕ.

The equation of Cq is

$$\frac{x}{a\sin\frac{\theta+\phi}{2}}=-\frac{y}{b\cos\frac{\theta+\phi}{2}},$$

and where it cuts the ellipse

$$x^2\left(\frac{1}{a^2}+\frac{1}{a^2}\cot^2\frac{\theta+\phi}{2}\right)=1.$$

$$\therefore\ x^2 = a\sin^2\frac{\theta+\phi}{2}.$$

$$\therefore\ Cq^2 = a^2\sin^2\frac{\theta+\phi}{2}+b^2\cos^2\frac{\theta+\phi}{2}.$$

Also $$PQ^2 = a^2(\cos\theta-\cos\phi)^2+b^2(\sin\theta-\sin\phi)^2$$

$$= 4\sin^2\frac{\theta-\phi}{2}\left|a^2\sin^2\frac{\theta+\phi}{2}+b^2\cos^2\frac{\theta+\phi}{2}\right|$$

$$= 4\sin^2\frac{\theta-\phi}{2}.Cq^2.$$

The equation of CD is $y = -\frac{b}{a}x\cot\theta$. Where it cuts PQ we have

$$\frac{x}{a}\left(\cos\frac{\theta+\phi}{2}-\cot\theta\sin\frac{\theta+\phi}{2}\right)=\cos\frac{\theta-\phi}{2}.$$

$$\therefore\ \frac{x}{a}\sin\frac{\theta-\phi}{2}=\cos\frac{\theta-\phi}{2}\sin\theta$$

and $$\frac{y}{b}\sin\frac{\theta-\phi}{2}=-\cos\frac{\theta-\phi}{2}\cos\theta.$$

Therefore

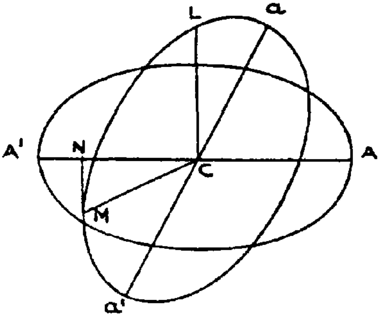
$$PR^2 = a^2\left(\cos\theta-\frac{\cos\frac{\theta-\phi}{2}\sin\theta}{\sin\frac{\theta-\phi}{2}}\right)^2+b^2\left(\sin\theta+\frac{\cos\frac{\theta-\phi}{2}\cos\theta}{\sin\frac{\theta-\phi}{2}}\right)^2.$$

$$\therefore\ PR^2\sin^2\frac{\theta-\phi}{2}=a^2\sin^2\frac{\theta+\phi}{2}+b^2\cos^2\frac{\theta+\phi}{2}=Cq^2.$$

$$\therefore\ PQ^2.PR^2 = 4Cq^4 \quad\text{and}\quad PQ.PR = 2Cq^2.$$

Ex. 6. The ellipse $b^2x^2+a^2y^2=a^2b^2$ is rotated about its centre in its own plane. Prove that the locus of the extremities of the diameter conjugate to the axis of y in the various positions of the ellipse is the curve whose equation is

$$x^2(a^2+b^2-x^2-y^2) = a^2b^2. \qquad \text{[O.]}$$



In the diagram the ellipse has been rotated through an angle θ. CL is its intercept on the axis CB of the ellipse in its original position. CM is the conjugate semidiameter to CL. MN is drawn perpendicular to the axis ACA'.

Then we have $\qquad CL^2+CM^2 = a^2+b^2$

and $\qquad CL.CM \sin \angle LCM = ab.$

Hence, if the coordinates of M are (x, y),

$$CL^2+x^2+y^2 = a^2+b^2.$$

Also $\qquad x = CM \cos \angle MCN = CM \sin \angle MCL.$

$$\therefore\ CL = \frac{ab}{x}.$$

Hence $\qquad a^2+b^2-x^2-y^2 = CL^2 = \dfrac{a^2b^2}{x^2}.$

And the required locus is

$$x^2(a^2+b^2-x^2-y^2) = a^2b^2.$$

Ex. 7. A circle touches an ellipse $b^2x^2+a^2y^2 = a^2b^2$ at a fixed point whose eccentric angle is α; it cuts it again in two points

Q and R. Show that, whatever the radius of the circle, the chord QR is fixed in direction.

If the line through the centre C parallel to QR cuts the tangent at Q in T, show that the equation of the locus of T is

$$b^2x^2+a^2y^2 = 4x^2y^2\operatorname{cosec}^2 2\alpha.$$

Suppose Q and R are the points ϕ_1, ϕ_2 respectively. Then

$$2\alpha+\phi_1+\phi_2 = 2n\pi.$$

$$\therefore\ \phi_1+\phi_2 = 2(n\pi-\alpha).$$

The gradient of the chord joining Q and R is

$$-\frac{b}{a}\cot\frac{\phi_1+\phi_2}{2} = \frac{b}{a}\cot\alpha,$$

which is constant. $\therefore$ QR is fixed in direction.

The line through the centre parallel to QR is

$$\frac{x}{a}\cos\alpha-\frac{y}{b}\sin\alpha = 0.$$

Where it cuts the tangent at ϕ_1, viz.

$$\frac{x}{a}\cos\phi_1+\frac{y}{b}\sin\phi_1 = 1,$$

$$\frac{x}{a}\sin(\alpha+\phi_1) = \sin\alpha, \qquad \frac{y}{b}\sin(\alpha+\phi_1) = \cos\alpha.$$

$$\therefore\ \frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{1}{\sin^2(\alpha+\phi_1)},$$

also
$$\frac{4x^2y^2}{a^2b^2} = \frac{4\sin^2\alpha\cos^2\alpha}{\sin^2(\alpha+\phi_1)} = \frac{\sin^2 2\alpha}{\sin^2(\alpha+\phi_1)}.$$

$$\therefore\ b^2x^2+a^2y^2 = 4x^2y^2\operatorname{cosec}^2 2\alpha$$

is the required locus.

Ex. 8. CL, CM are two semidiameters of an ellipse and Cl, Cm are two semidiameters conjugate to them respectively.

Show that LM is either parallel to lm or that LM and lm are parallel to two diameters which are conjugate.

Let L and M be the points θ, ϕ on the ellipse

$$b^2x^2+a^2y^2 = a^2b^2.$$

Then l may be the point $\frac{\pi}{2}+\theta$ or $\frac{3\pi}{2}+\theta$, and m the point $\frac{\pi}{2}+\phi$ or $\frac{3\pi}{2}+\phi$.

Taking l as $\frac{\pi}{2}+\theta$ and m as $\frac{\pi}{2}+\phi$, the equation of lm is

$$\frac{x}{a}\cos\tfrac{1}{2}(\pi+\theta+\phi)+\frac{y}{b}\sin\tfrac{1}{2}(\pi+\theta+\phi) = \cos\frac{\theta-\phi}{2},$$

whose gradient is $-\frac{b}{a}\cot\tfrac{1}{2}(\pi+\theta+\phi) = \frac{b}{a}\tan\frac{\theta+\phi}{2}$.

Taking l as $\frac{3\pi}{2}+\theta$ and m as $\frac{\pi}{2}+\phi$, the equation of lm is

$$\frac{x}{a}\cos\tfrac{1}{2}(2\pi+\theta+\phi)+\frac{y}{b}\sin\tfrac{1}{2}(2\pi+\theta+\phi) = -\sin\frac{\theta-\phi}{2},$$

whose gradient is $-\frac{b}{a}\cot\left(\pi+\frac{\theta+\phi}{2}\right) = -\frac{b}{a}\cot\frac{\theta+\phi}{2}$, and the gradient of LM is $-\frac{b}{a}\cot\frac{\theta+\phi}{2}$.

In the first case the product of the gradients of LM and lm is $-\frac{b^2}{a^2}$, i.e. they are parallel to conjugate diameters.

In the second case the gradients are the same, i.e. LM and lm are parallel.

Ex. 9. **If a tangent to an ellipse at any point cuts the director circle in H and K, then CH, CK will be conjugate semidiameters of the ellipse.**

The equation of the tangent at a point ϕ being

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi = 1,$$

the equation of the lines CH, CK joining the centre to the points where this tangent cuts the director circle is

$$x^2+y^2 = (a^2+b^2)\left(\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi\right)^2,$$

i.e. $$b^2x^2(a^2\sin^2\phi-b^2\cos^2\phi)-a^2y^2(a^2\sin^2\phi-b^2\cos^2\phi)$$
$$-2ab(a^2+b^2)\sin\phi\cos\phi xy = 0.$$

If these lines are $y = mx$, $y = m'x$, then

$$mm' = -\frac{b^2}{a^2},$$

i.e. the lines are conjugate.

Ex. 10. The base AB of a triangle is fixed and its sides are parallel to conjugate diameters of an ellipse. Find the locus of its vertex.

Let O be the middle point of AB.

Take AOB and a line through O perpendicular to it as axes of coordinates.

Let A, B be the points $(c,0)$, $(-c,0)$.

The equation of any line through A is

$$y = m(x-c).$$

That of a line through B parallel to the diameter of the ellipse which is conjugate to $y = mx$ is

$$y = -\frac{b^2}{a^2m}(x+c).$$

$\therefore$ the locus of the vertex is

$$y^2 = -\frac{b^2}{a^2}(x^2-c^2),$$

i.e. $$a^2y^2+b^2x^2 = b^2c^2,$$

which is the equation of an ellipse concentric with and similar to the given ellipse.

Ex. 11. To find the equation of a conic when the axes of coordinates are the tangent and normal at any point on it.

Since the origin is on the curve, its equation can be written

$$ax^2+2hxy+by^2+2gx+2fy = 0.$$

The line $y = 0$ touches the curve at the origin, $\therefore$ both values of x given by $ax^2+2gx = 0$ are zero. $\therefore\ g = 0$.

Hence the most general form of the equation of a conic referred to a tangent and its corresponding normal as axes is

$$ax^2+2hxy+by^2+2fy = 0.$$

Ex. 12. If two concentric ellipses have the same axes and their points of intersection are the extremities of conjugate diameters of one of them, find a relation between their major and minor semi-axes.

Suppose the ellipses are $\dfrac{x^2}{a_1^2}+\dfrac{y^2}{b_1^2} = 1$ and $\dfrac{x^2}{a_2^2}+\dfrac{y^2}{b_2^2} = 1$.

Where these intersect, we have

$$x^2\left(\frac{1}{a_1^2}-\frac{1}{a_2^2}\right) = -y^2\left(\frac{1}{b_1^2}-\frac{1}{b_2^2}\right).$$

If these two lines are conjugate diameters of the first ellipse, the product of their gradients must be $-\dfrac{b_1^2}{a_1^2}$.

$$\therefore\ \frac{\dfrac{1}{a_1^2}-\dfrac{1}{a_2^2}}{\dfrac{1}{b_1^2}-\dfrac{1}{b_2^2}} = -\frac{b_1^2}{a_1^2};$$

whence

$$\frac{a_1^2}{a_2^2}+\frac{b_1^2}{b_2^2} = 2$$

is the required relation.

Ex. 13. Find the locus of points from which two tangents can be drawn to an ellipse which will cut the axes in points which are concyclic.

The equation of the tangents from (h, k) to the ellipse $a^2x^2+b^2y^2 = a^2b^2$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right) = \left(\frac{xh}{a^2}+\frac{yk}{b^2}-1\right)^2.$$

Where this cuts $x = 0$,

$$\frac{y^2}{b^2}\left(\frac{h^2}{a^2}-1\right)+\frac{2yk}{b^2}-\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right) = 0.$$

If the roots of this equation are y_1, y_2, then

$$y_1\,y_2 = -\frac{a^2b^2}{h^2-a^2}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right).$$

Similarly, where the tangents cut $y = 0$,

$$x_1\,x_2 = -\frac{a^2b^2}{k^2-b^2}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right).$$

If $x_1\,x_2 = y_1\,y_2$, i.e. if the four points are concyclic,

$$h^2-a^2 = k^2-b^2,$$

and the required locus is

$$x^2-y^2 = a^2-b^2.$$

Ex. 14. Find the locus of the intersection of normals at the ends of conjugate diameters of the ellipse $b^2x^2+a^2y^2 = a^2b^2$.

The equation of the normal at the point ϕ is

$$\frac{ax}{\cos\phi}-\frac{by}{\sin\phi} = a^2-b^2,$$

and that of the normal at the point $\frac{\pi}{2}+\phi$ is

$$\frac{ax}{\sin\phi}+\frac{by}{\cos\phi} = -(a^2-b^2);$$

whence

$$\frac{ax}{\sin\phi-\cos\phi}=-\frac{by}{\sin\phi+\cos\phi}=(a^2-b^2)\sin\phi\cos\phi.$$

$$\therefore\ a^2x^2+b^2y^2$$

$$=(a^2-b^2)^2\sin^2\phi\cos^2\phi[(\sin\phi-\cos\phi)^2+(\sin\phi+\cos\phi)^2]$$

$$=2(a^2-b^2)^2\sin^2\phi\cos^2\phi$$

and $$a^2x^2-b^2y^2=-4(a^2-b^2)^2\sin^3\phi\cos^3\phi.$$

$$\therefore\ 2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2.$$

Ex. 15. Find, in terms of ϕ, the coordinates of the point in which the normal at the point ϕ on the ellipse $b^2x^2+a^2y^2=a^2b^2$ again cuts the curve.

The equation of the normal to the ellipse at the point ϕ is

$$\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2=c^2,\quad\text{suppose.}$$

In this equation put $x=a\cos\theta$, $y=b\sin\theta$, then

$$\frac{a^2\cos\theta}{\cos\phi}-\frac{b^2\sin\theta}{\sin\phi}=c^2,\tag{1}$$

i.e. $$\left(\frac{a^2\cos\theta}{\cos\phi}-c^2\right)^2=\frac{b^4\sin^2\theta}{\sin^2\phi}=\frac{b^4(1-\cos^2\theta)}{\sin^2\phi}.$$

$$\therefore\ \left(\frac{a^4}{\cos^2\phi}+\frac{b^4}{\sin^2\phi}\right)\cos^2\theta-\frac{2a^2c^2}{\cos\phi}\cos\theta+c^4-\frac{b^4}{\sin^2\phi}=0,$$

which is a quadratic equation in $\cos\theta$ giving the two values of $\cos\theta$ for the two points in which the normal at ϕ cuts the ellipse.

$\text{Cos}\,\phi$ must be one root of this equation: suppose that $\cos\alpha$ is the other root, then

$$\cos\phi\cos\alpha=\frac{(c^4\sin^2\phi-b^4)\cos^2\phi}{a^4\sin^2\phi+b^4\cos^2\phi}.$$

$$\therefore\ a\cos\alpha = \frac{a\cos\phi(c^4\sin^2\phi - b^4)}{a^4\sin^2\phi + b^4\cos^2\phi}.$$

Similarly, by making equation (1) a quadratic in $\sin\theta$ instead of in $\cos\theta$, we should get

$$b\sin\alpha = \frac{b\sin\phi(c^4\cos^2\phi - a^4)}{a^4\sin^2\phi + b^4\cos^2\phi},$$

so that the coordinates of the point α in which the normal at the point ϕ cuts the curve again are determined in terms of ϕ.

EXAMPLES 13

1. If CP and CD are conjugate semidiameters of an ellipse, prove that the sum of the squares of the abscissae of P and D is a^2, and that the sum of the squares of their ordinates is b^2.

2. In the ellipse $4x^2+9y^2 = 36$, what are the equations of the diameters conjugate to (1) $y = x$, (2) $y = -x$, and (3) $y = 2x$?

3. CP and CD are conjugate semidiameters of the ellipse $b^2x^2+a^2y^2 = a^2b^2$. The eccentric angle of P is ϕ. Prove that $CP^2 = a^2(1-e^2\sin^2\phi)$ and $CD = a^2(1-e^2\cos^2\phi)$.

4. Show that in the ellipse $\frac{x^2}{64}+\frac{y^2}{36} = 1$, $y = 3x$ and $16y+3x = 0$ are conjugate diameters.

5. A pair of conjugate semidiameters CP, CD of an ellipse are produced to meet a directrix in L and M. Prove that the focus of the ellipse is the orthocentre of the triangle CLM.

6. The eccentric angle of a point P on an ellipse

$$b^2x^2+a^2y^2 = a^2b^2$$

is $\tan^{-1}\frac{b^2}{a^2}$. CP and CD are conjugate semidiameters on the same side of the major axis. DE is drawn perpendicular to the major axis to meet the curve again in E. Show that EP is the normal at P.

7. Prove that the major and minor axes of an ellipse are the only pair of perpendicular conjugate diameters.

8. Find the coordinates of the points where the lines $y = mx$, $my+x = 0$ cut the ellipse $b^2x^2+a^2y^2 = a^2b^2$. Hence show that

the sum of the squares of the reciprocals of two perpendicular diameters of an ellipse is constant.

9. CP and CD are a pair of conjugate semidiameters of an ellipse. Find the locus of the middle point of PD.

10. CP and CD are a pair of conjugate semidiameters of an ellipse. Find the locus of the pole of PD.

11. CP and CD are a pair of conjugate semidiameters of an ellipse. CY is drawn perpendicular to PD. Find the locus of Y.

12. If the normal at P to an ellipse meets the major axis in G and the diameter conjugate to CP in F, prove that $PG.PF = b^2$.

13. If P is a point on an ellipse whose eccentric angle is ϕ, and CD is the semidiameter conjugate to CP, find the coordinates of the point of intersection of the normals at P and D to the ellipse.

14. CP and CD are conjugate semidiameters of an ellipse and the normals at P and D meet at K. Show that CK is perpendicular to PD.

15. If the normal at a point P of an ellipse cuts the minor axis in g, and gS produced cuts the tangent at the vertex in Q, show that $gQ = Pg$.

16. If the tangents at the ends of conjugate semidiameters of an ellipse $b^2x^2+a^2y^2 = a^2b^2$ meet in a point (h, k) and the normals at the same points meet in a point (p, q), prove that

$$hq : kp = a^2 : b^2.$$

17. CP, CD are conjugate semidiameters of an ellipse. On PD produced a point Q is taken so that $QD = 2PD$. Find the locus of Q.

18. CP, CD are conjugate semidiameters of the ellipse $b^2x^2+a^2y^2 = a^2b^2$. PD is inclined to the major axis at an angle θ. The length of the perpendicular from the centre on PD is d. Prove that $d^2 = \frac{1}{2}(a^2\cos^2\theta+b^2\sin^2\theta)$.

19. Prove that the sum of the lengths of two focal chords of an ellipse drawn parallel to a pair of conjugate diameters is constant.

20. Show that the product of the perpendiculars from the points $(0, \pm be)$ to the tangent at any point P of an ellipse $b^2x^2+a^2y^2=a^2b^2$ is $a^2b^2 . \dfrac{CP^2}{CD^2}$, where CD is the semidiameter conjugate to CP.

21. CP, CD are conjugate semidiameters of an ellipse. Points Q_1, Q_2 are taken on the normal at P such that $PQ_1 = PQ_2 = CD$. Prove that $CQ_1 = a-b$ and $CQ_2 = a+b$.

22. CP, CD are a pair of conjugate semidiameters of an ellipse. H is a point on the normal at P such that $PH = k.CD$. Find the locus of H. Use this result to find the locus of the centre of the inscribed circle of the triangle SPS', S and S' being the foci of the ellipse.

23. Show that the lines joining the centre of an ellipse $b^2x^2+a^2y^2 = a^2b^2$ to its points of intersection with the line $y = x+\sqrt{\left(\frac{a^2+b^2}{2}\right)}$ will be a pair of conjugate diameters of the ellipse.

24. Prove that the polar of any point on a diameter of an ellipse is parallel to the conjugate diameter.

25. Show that if a semidiameter CP of an ellipse be conjugate to the normal at another point Q on it, then CQ will be conjugate to the normal at P.

26. Find the locus of the middle points of the portions of tangents to an ellipse included between the axes.

27. Find the locus of the middle points of chords of an ellipse drawn through an extremity of its minor axis.

28. Find the locus of the middle points of chords of an ellipse which subtend a right angle at its centre.

29. Find the locus of the middle points of chords of an ellipse which are of a fixed length $2l$.

30. If L be an extremity of a latus rectum of an ellipse and CD the semidiameter conjugate to CL, find the angle between CD and CL.

31. The perpendiculars from the foci of an ellipse on a pair of conjugate semidiameters meet at a point Q. Find the locus of Q.

32. Prove that if PP' is a diameter of an ellipse, the line joining the points whose eccentric angles are $\angle CSP$ and $\angle CSP'$ will pass through the focus S.

33. Find the angle between the tangents to an ellipse

$$b^2x^2+a^2y^2 = a^2b^2$$

from a point (h, k) outside it.

34. Find the angle between the tangents to an ellipse

$$b^2x^2+a^2y^2 = a^2b^2$$

from a point $(a\cos\theta, a\sin\theta)$ on the auxiliary circle.

35. From a point T on the director circle of an ellipse tangents TP, TQ are drawn to the ellipse. Find the locus of the middle point of PQ.

36. From the point $(6, \sqrt{5})$ two tangents are drawn to the ellipse $x^2+9y^2 = 9$. Find the area of the triangle TPQ.

37. In the ellipses $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{a}+\dfrac{y^2}{b} = a+b$, a tangent to the former intersects the latter in points P and Q. Prove that the tangents at P and Q meet on the director circle of the second ellipse.

38. The tangents from a point to an ellipse cut a fixed tangent in two points H and K, such that $\angle HCK$ is a right angle. Find the locus of the point.

39. Tangents are drawn to an ellipse at the extremities P, D of a pair of conjugate semidiameters and they meet in T. Prove that $ST.S'T$ must be less than a^2+b^2.

40. A common tangent is drawn to the ellipse $b^2x^2+a^2y^2 = a^2b^2$ and the concentric circle $x^2+y^2 = c^2$. If $a > c > b$, prove that the eccentric angle of the point of contact with the ellipse is given by

$$\tan\phi = \frac{b}{a}\sqrt{\left(\frac{a^2-c^2}{c^2-b^2}\right)}.$$

41. Find the locus of the point of intersection of two tangents to an ellipse which always cut off an intercept of fixed length $2l$ on the major axis.

42. Show that the locus of the pole of a chord of an ellipse which subtends a right angle at the focus $(ae, 0)$ is

$$x^2(1-2e^2)+y^2+2aex = a^2(2-e^2).$$

43. PQ is a chord of an ellipse and p and q are the points on

the auxiliary circle corresponding to PQ. If $\angle pCq$ is constant, find the envelope of PQ.

44. If CP, CD be conjugate semidiameters of an ellipse, find the envelope of PD.

45. From any point T on the director circle of the ellipse $b^2x^2+a^2y^2 = a^2b^2$, tangents TP, TQ are drawn to the ellipse. Find the envelope of PQ.

46. Show that the envelope of the polar of any point on the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4} = \frac{1}{a^2}$ with regard to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is the auxiliary circle of the latter ellipse.

47. Q is a point on the normal at P to an ellipse such that CP, CQ make equal angles with the major axis. Show that $PQ : CD$ is a constant ratio, where CD is the semidiameter conjugate to CP.

48. PQ is a focal chord of an ellipse whose middle point is V; the normals at P and Q meet in O. Prove that OV is parallel to the major axis.

49. Find the locus of the centre of a circle which passes through a fixed point and touches a fixed circle internally.

50. In the ellipse $\frac{x^2}{(\sqrt{2}+1)^2} + y^2 = 1$ prove that the circle whose diameter joins the ends of two conjugate semidiameters of the ellipse will touch the ellipse.

51. PCP', DCD' are conjugate diameters of an ellipse $b^2x^2+a^2y^2 = a^2b^2$ and ϕ is the eccentric angle of the point P. What is the eccentric angle of the point in which the circle circumscribing the triangle $PP'D$ again cuts the ellipse?

52. PSQ is a focal chord of an ellipse and the tangents at P and Q meet at T. If S' be the other focus, prove that TS' passes through the point of intersection of the normals at P and Q.

53. CP, CD are two conjugate semidiameters of an ellipse. The circle whose diameter is PD touches the ellipse. Find the eccentricity of the ellipse.

54. The normal at P meets the major axis in G, and through G a line is drawn parallel to the minor axis, meeting CP in Q. Show that the locus of Q is $b^2x^2+a^2y^2 = a^2b^2e^4$.

55. Prove that the coordinates of the orthocentre of the triangle formed by the three points on an ellipse $b^2x^2+a^2y^2 = a^2b^2$ whose eccentric angles are θ_1, θ_2, θ_3 are

$$\frac{1}{2}\left[\frac{a^2+b^2}{a}(\cos\theta_1+\cos\theta_2+\cos\theta_3)-\frac{a^2-b^2}{a}\cos(\theta_1+\theta_2+\theta_3)\right]$$

and $$\frac{1}{2}\left[\frac{a^2+b^2}{b}(\sin\theta_1+\sin\theta_2+\sin\theta_3)-\frac{a^2-b^2}{b}\sin(\theta_1+\theta_2+\theta_3)\right].$$

56. A triangle inscribed in an ellipse has its orthocentre at the centre of the ellipse. Prove that the normals at its vertices are concurrent.

57. Three normals are drawn to the ellipse $b^2y^2+a^2x^2 = a^2b^2$ from any point on the curve. Show that the locus of the centroid of the triangle formed by their feet is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{(a^2+b^2)^2}{9(a^2-b^2)^2}.$$

58. From the vertex of an ellipse a perpendicular is drawn to the normal at a point whose eccentric angle is ϕ. Show that it meets the ellipse again in a point whose eccentric angle is $\frac{\phi}{2}$.

59. If from the vertex of an ellipse perpendiculars are drawn to four concurrent normals, these perpendiculars will meet the curve again in four concyclic points.

60. From a point P on an ellipse $b^2x^2+a^2y^2 = a^2b^2$, three normals other than the normal at the point are drawn. Verify that their feet lie upon the circle

$$x^2+y^2-\frac{b^2}{a}x\cos\phi-\frac{a^2}{b}y\sin\phi-a^2-b^2 = 0,$$

where ϕ is the eccentric angle of the point P.

61. Show that the feet of the normals of the ellipse

$$b^2x^2+a^2y^2 = a^2b^2$$

which pass through though the point (h, k) lie on the curve

$$xy(a^2-b^2)+ha^2y-kb^2x = 0.$$

62. Show that a circle goes through the middle points of the chords joining any pair of conormal points on an ellipse. Also, prove that this circle passes through the point of concurrence of the normals and through the centre of the ellipse.

Formulae of Chapter VI for reference

1. $CS = ae$; $CX = \dfrac{a}{e}$; $b^2 = a^2(1-e^2)$; $SP+S'P = 2a$.

2. Latus rectum is: $\dfrac{2b^2}{a} = 2a(1-e^2)$.

3. Equation of a chord:

$$\frac{x}{a}\cos\frac{\phi+\phi'}{2}+\frac{y}{b}\sin\frac{\phi+\phi'}{2} = \cos\frac{\phi-\phi'}{2}.$$

4. Equations of a tangent: $\dfrac{x}{a}\cos\phi+\dfrac{y}{b}\sin\phi = 1.$

$$\frac{xx'}{a^2}+\frac{yy'}{b^2} = 1.$$

$$y = mx\pm\sqrt{(a^2m^2+b^2)}.$$

5. Equation of a normal: $\dfrac{ax}{\cos\phi}-\dfrac{by}{\sin\phi} = a^2-b^2 = a^2e^2.$

6. Two tangents meet at the point: $\dfrac{a\cos\frac{\phi+\phi'}{2}}{\cos\frac{\phi-\phi'}{2}}, \dfrac{b\sin\frac{\phi+\phi'}{2}}{\cos\frac{\phi-\phi'}{2}}.$

7. Equation of the polar of (h, k): $\dfrac{xh}{a^2}+\dfrac{yk}{b^2} = 1.$

8. Polar equation with respect to focus: $\dfrac{l}{r} = 1-e\cos\theta.$

Polar equation with respect to centre: $\dfrac{1}{r^2} = \dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}.$

9. Polar equation of tangent: $\dfrac{l}{r} = \cos(\theta-\alpha)-e\cos\theta.$

10. *Equation of auxiliary circle:* $x^2+y^2 = a^2.$

11. *Equation of director circle:* $x^2+y^2 = a^2+b^2.$

12. $y = mx$, $y = m'x$ are conjugate if: $\quad \mathbf{mm' = -\frac{b^2}{a^2}}$.

13. If CP, CD are conjugate semidiameters:

$$\mathbf{P \text{ is the point } a\cos\varphi, b\sin\varphi,}$$

$$\mathbf{D \text{ is the point } -a\sin\varphi, b\cos\varphi,}$$

$$\mathbf{CP^2+CD^2 = a^2+b^2,}$$

$$\mathbf{\Delta CPD = \tfrac{1}{2}ab.}$$

14. If $\phi_1, \phi_2, \phi_3, \phi_4$ are conormal points:

$$\mathbf{\varphi_1+\varphi_2+\varphi_3+\varphi_4 = (2n+1)\pi.}$$

15. If $\phi_1, \phi_2, \phi_3, \phi_4$ are concyclic points:

$$\mathbf{\varphi_1+\varphi_2+\varphi_3+\varphi_4 = 2n\pi.}$$

16. Equation of tangents from (h, k):

$$\mathbf{\left(\frac{xh}{a^2}+\frac{yk}{b^2}-1\right)^2 = \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right).}$$

VII

THE HYPERBOLA

1. Definition and equation of the hyperbola

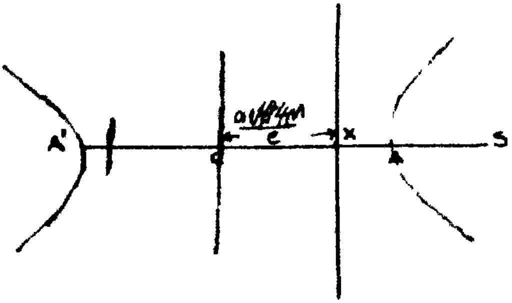
If a plane is drawn cutting all the generators of both branches of a cone, it will, in general, intersect the conical surface in a curve called a *Hyperbola*.

A hyperbola is also defined as the locus of a point which moves so that its distance from a fixed point called the *Focus* is always a constant ratio, greater than unity, of its distance from a fixed straight line called the *Directrix*.

As in the case of the ellipse, this ratio is known as the *Eccentricity* of the hyperbola, and is usually denoted by e, where $e > 1$, instead of being < 1, as with the ellipse.

Let S be a focus of a hyperbola whose eccentricity is e.

Draw SX, perpendicular to the directrix and produce it indefinitely in both directions.



Divide SX externally and internally, in the ratio $e : 1$ at the points A, A'. Both A and A' will be points on the hyperbola since they satisfy the definition of the locus, since

$$SA : AX = e : 1 = SA' : A'X.$$

Bisect AA' at C and let $CA = CA' = a$.

We shall now prove that (1) $CX = \frac{a}{e}$, (2) $CS = ae$.

(1) $SA'-SA = e(A'X-AX) = e(CA'+CX-CA+CX)$.

$$\therefore\ AA' = 2eCX \quad \text{and} \quad CX = \frac{a}{e}.$$

(2) $SA'+SA = SC+CA'+SC-CA = 2SC$.

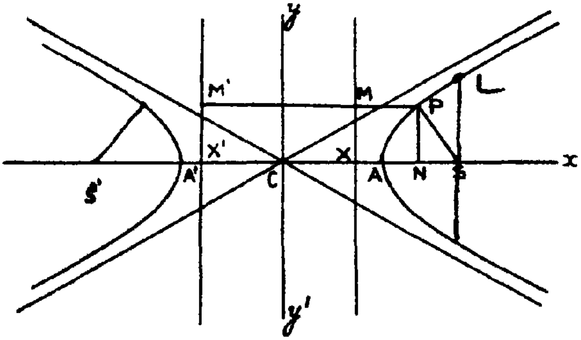
$$\therefore\ 2SC = SA'+SA = e(A'X+AX) = eAA'.$$

$$\therefore\ CS = ae.$$

Take the point C as origin, the line $AXCA'$ as axis of x, and a line through C perpendicular to it as axis of y.

Let P be any point on the hyperbola whose coordinates are (x, y).

Draw PN and PM perpendicular to the x-axis and the directrix respectively and join SP.



By definition, $SP^2 = e^2PM^2 = e^2NX^2$.

$$\therefore\ SN^2+PN^2 = e^2NX^2.$$

Now $SN = CS-CN = ae-x$

and $NX = CN-CX = x-\frac{a}{e}$.

$$\therefore\ (ae-x)^2+y^2 = e^2\left(x-\frac{a}{e}\right)^2.$$

$$\therefore\ x^2(e^2-1)-y^2 = a^2(e^2-1),$$

i.e.
$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)} = 1.$$

Let $\mathbf{b^2 = a^2(e^2-1)}$, and this equation becomes

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}-\frac{\mathbf{y}^2}{\mathbf{b}^2} = \mathbf{1}.$$

From the form of this equation certain facts can be deduced by inspection about the hyperbola it represents.

(1) As $-x$, $-y$ can be substituted in the equation for x and y without altering its form, we infer that the curve is symmetrical about the origin and that this point is its centre.

(2) As $-x$ can be put for x, leaving y unchanged, or $-y$ for y, leaving x unchanged, without altering the form of the equation, we infer that the curve is symmetrical about either axis.

(3) Since $y^2 = b^2\left(\frac{x^2}{a^2}-1\right)$, it follows that, for any value of x lying between $+a$ and $-a$, y is imaginary; but that if x does not lie between these limits, y is real.

This demonstrates that no part of the curve can lie between the two lines $x = a$ and $x = -a$.

(4) Since $\frac{x^2}{a^2} = \left(1+\frac{y^2}{b^2}\right)$, there is no limit to the magnitude of the real values x may have for either positive or negative real values of y.

Hence we conclude that the curve will consist of two symmetrical branches, each going off to infinity in two directions, as in the diagram on the previous page.

The symmetry of the curve about the y-axis also indicates that there will be two foci, each of them with a directrix corresponding to it.

The line ACA' is called the *Transverse Axis* of the hyperbola rather than its major axis.

The line through C perpendicular to ACA' has been taken as the y-axis whose equation is $x = 0$. Where $x = 0$ cuts the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have $y^2 = -b^2$. $\therefore\ y = \pm b\sqrt{(-1)}$.

Thus the axis of the hyperbola perpendicular to the transverse axis is real in direction but imaginary in magnitude.

This axis will be subsequently referred to as the *Conjugate Axis* of the hyperbola and its length as $2b$, although the true length of the semi-axis perpendicular to the transverse axis is $b\sqrt{(-1)}$.

The points A and A' are the *Vertices* of the hyperbola.

Its *Latus Rectum*, i.e. the chord through a focus perpendicular to the transverse axis, is equal to

$$2eSX = 2e(SC - CX) = 2e\left(ae - \frac{a}{e}\right) = 2a(e^2 - 1)$$
$$= \frac{2b^2}{a}.$$

If PM be produced to cut the directrix corresponding to the focus S', in M', then

$$S'P - SP = e(M'P - MP) = eMM' = 2eCX$$
$$= 2a.$$

Hence the difference of the focal radii of any point on a hyperbola is equal to the transverse axis.

2. Parametric coordinates

The equation of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is satisfied by $x = a\sec\phi$, $y = b\tan\phi$, for any value of ϕ. These may therefore be taken as the coordinates of any point on the hyperbola, and the parameter as ϕ.

The equation of the chord joining the two points ϕ, ϕ' is

$$\frac{x-a\sec\phi}{a(\sec\phi-\sec\phi')}=\frac{y-b\tan\phi}{b(\tan\phi-\tan\phi')},$$

i.e. $$\frac{x-a\sec\phi}{a(\cos\phi'-\cos\phi)}=\frac{y-b\tan\phi}{b(\sin\phi\cos\phi'-\cos\phi\sin\phi')},$$

i.e. $$\frac{x-a\sec\phi}{2a\sin\frac{\phi+\phi'}{2}\sin\frac{\phi-\phi'}{2}}=\frac{y-b\tan\phi}{2b\sin\frac{\phi-\phi'}{2}\cos\frac{\phi-\phi'}{2}},$$

i.e. $$\frac{x}{a}\cos\frac{\phi-\phi'}{2}-\frac{y}{b}\sin\frac{\phi+\phi'}{2}=\frac{\cos\frac{\phi-\phi'}{2}}{\cos\phi}-\frac{\sin\phi\sin\frac{\phi+\phi'}{2}}{\cos\phi}.$$

But

$$\cos\frac{\phi-\phi'}{2}-\sin\phi\sin\frac{\phi+\phi'}{2}=\cos\frac{\phi-\phi'}{2}-2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\sin\frac{\phi+\phi'}{2}$$

$$=\cos\frac{\phi}{2}\cos\frac{\phi'}{2}\left(1-2\sin^2\frac{\phi}{2}\right)-\sin\frac{\phi}{2}\sin\frac{\phi'}{2}\left(2\cos^2\frac{\phi}{2}-1\right)$$

$$=\cos\phi\cos\frac{\phi+\phi'}{2}.$$

$\therefore$ the equation of the chord is

$$\frac{x}{a}\cos\frac{\phi-\phi'}{2}-\frac{y}{b}\sin\frac{\phi+\phi'}{2}=\cos\frac{\phi+\phi'}{2}.$$

The equation of the tangent at the point ϕ is obtainable from this equation by putting $\phi'=\phi$, and is

$$\frac{x}{a}-\frac{y}{b}\sin\phi=\cos\phi.$$

The equation of the normal is

$$a\sin\phi(x-a\sec\phi)+b(y-b\tan\phi)=0,$$

i.e. $$ax\sin\phi+by=(a^2+b^2)\tan\phi.$$

These forms of the equations, however, are not very often employed because, as we shall see later, in dealing with the majority of problems connected with the hyperbola, its axes are not the most convenient to take as axes of coordinates.

Here it may be mentioned for the benefit of students who have an acquaintance with hyperbolic functions that

$$x = a\cosh\phi,\quad y = b\sinh\phi$$

would satisfy the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ for all values of ϕ, and that these might be taken as the coordinates of any point on it.

The equation of the tangent at the point $(a\cosh\phi, b\sinh\phi)$ is

$$\frac{x}{a}\cosh\phi-\frac{y}{b}\sinh\phi = 1.$$

3. Corresponding formulae

Practically all the results obtained in connexion with the ellipse in the previous section can be applied to the hyperbola merely by putting $-b^2$ for b^2 in the corresponding equation for the ellipse. Thus:

(1) The equation of the tangent at the point (x', y') is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2} = 1.$$

(2) The equation of the normal at the point (x', y') is

$$\frac{a^2x}{x'}+\frac{b^2y}{y'} = a^2+b^2.$$

(3) The two tangents to the hyperbola of gradient m are

$$y = mx\pm\sqrt{(a^2m^2-b^2)}.$$

(4) The line $lx+my+n = 0$ will touch the hyperbola if

$$a^2l^2-b^2m^2 = n^2.$$

(5) The condition for $y = mx$, $y = m'x$ to be conjugate diameters is

$$mm' = \frac{b^2}{a^2}.$$

(6) The equation of the tangents from a point (h, k) is

$$\left(\frac{xx'}{a^2}-\frac{yy'}{b^2}-1\right)^2 = \left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-1\right).$$

(7) The equation of the polar of (h, k) is

$$\frac{xh}{a^2}-\frac{yk}{b^2} = 1.$$

(8) The equation of the director circle is $x^2+y^2 = a^2-b^2$.

(9) Generally, all the geometrical properties proved in the last section for an ellipse are also true for the hyperbola.

(10) Polar equation $\dfrac{l}{r} = 1-e\cos\theta$.

These results could all be proved in exactly the same way as they were in the case of the ellipse.

4. Asymptotes

An *Asymptote* of a curve is a straight line which meets it at two points at infinity, but which is not altogether at infinity.

The polar equation of the hyperbola, with its centre as the pole, is

$$\frac{r^2\cos^2\theta}{a^2}-\frac{r^2\sin^2\theta}{b^2} = 1,$$

i.e.

$$r^2 = \frac{a^2b^2}{b^2\cos^2\theta-a^2\sin^2\theta}.$$

If $\tan\theta = \pm\dfrac{b}{a}$, it is clear that the values of r are infinite. If $a^2\sin^2\theta > b^2\cos^2\theta$, i.e. if $\tan\theta$ does not lie between $\dfrac{b}{a}$ and $-\dfrac{b}{a}$, r is imaginary, which shows that the hyperbola lies entirely between two lines drawn from the centre making angles of $\tan^{-1}\dfrac{b}{a}$ with the transverse axis.

The cartesian equation of the asymptotes can be obtained thus. We know that the equation $0x^2+0x+c = 0$ has two roots equal and infinite. This may be proved by putting $x = \frac{1}{y}$ in the equation $cx^2+0x+0 = 0$ which has two equal roots of zero. For the equation $0y^2+0y+c = 0$ thereby obtained must have as its equal roots the reciprocals of the equation whose two equal roots were zero, i.e. they are both infinite.

Consider the points in which the line $y = mx+c$ cuts the hyperbola whose equation is $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$.

For their abscissae, we have

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2} = 1,$$

i.e. $$x^2\left(\frac{1}{a^2}-\frac{m^2}{b^2}\right)-\frac{2mc}{b^2}x-\frac{c^2}{b^2}-1 = 0,$$

both of the roots of which are infinite if

$$mc = 0 \quad \text{and} \quad \frac{1}{a^2}-\frac{m^2}{b^2} = 0.$$

Hence, if $c = 0$ and $m = \pm\frac{b}{a}$, the line $y = mx+c$ will be an asymptote.

$\therefore$ **the equations of the asymptotes are $\mathbf{y = \pm\frac{b}{a}x}$, or written** as one equation $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 0$.

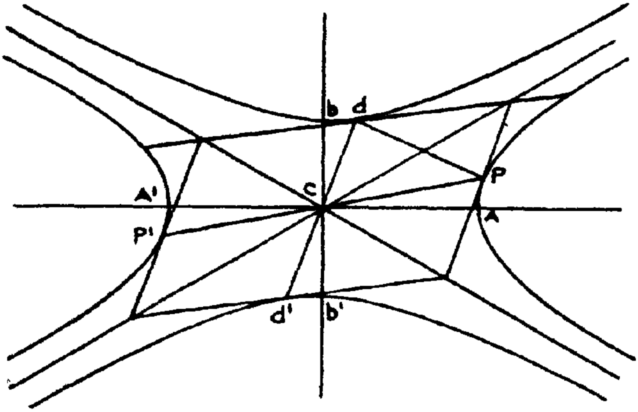
If in the above $m = \pm\frac{b}{a}$ but c is not zero, then the lines $y = \pm\frac{b}{a}x+c$ will cut the hyperbola in one point only at infinity, i.e. they will be parallel to the asymptotes.

5. The conjugate hyperbola

A hyperbola with the same centre and asymptotes as the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$, with its transverse axis lying along the conjugate axis of this hyperbola and of length $2b$, is called its *Conjugate Hyperbola.*

The equation of the conjugate hyperbola is $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ or

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.$$



The form of this equation shows that for any value of y lying between $+b$ and $-b$ the corresponding value of x is imaginary. Hence no part of the conjugate hyperbola can lie within the area enclosed by the lines $y = \pm b$.

Again, the polar equation of the conjugate hyperbola, with its centre as pole, is

$$r^2 = \frac{a^2b^2}{a^2\sin^2\theta - b^2\cos^2\theta},$$

which shows that r is imaginary if $\tan\theta$ lies between $\frac{b}{a}$ and $-\frac{b}{a}$.

Thus the conjugate hyperbola lies entirely in those two sections formed by the asymptotes in which the original hyperbola does not lie.

The condition for two diameters $y = mx$, $y = m'x$ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to be conjugate is $mm'=\frac{b^2}{a^2}$. But this condition would make them conjugate also for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$.

Hence all diameters which are conjugate with respect to a hyperbola are also conjugate with respect to its conjugate hyperbola, and vice versa.

Suppose that P is any point $(a\sec\phi, b\tan\phi)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The equation of CP is $\frac{x}{a\sec\phi}=\frac{y}{b\tan\phi}$, i.e. $y=\frac{bx\sin\phi}{a}$.

The equation of the conjugate diameter is $y=\frac{bx}{a\sin\phi}$.

Where this line cuts the hyperbola,

$$x^2\left(\frac{1}{a^2}-\frac{1}{a^2\sin^2\phi}\right)=1,$$

and the values of x are imaginary, i.e. the diameter conjugate to CP does not meet the hyperbola again in real points.

Suppose that it cuts its conjugate hyperbola in a point d. The abscissa of this point is given by

$$x^2\left(\frac{1}{a^2}-\frac{1}{a^2\sin^2\phi}\right)=-1;$$

i.e. it is the point $(a\tan\phi, b\sec\phi)$, and

$$Cd^2=a^2\tan^2\phi+b^2\sec^2\phi.$$

$$\therefore\ CP^2-Cd^2=(a^2-b^2)(\sec^2\phi-\tan^2\phi)=a^2-b^2.$$

The tangents at P and d to the hyperbola and its conjugate are respectively

$$\frac{x}{a}-\frac{y}{b}\sin\phi=\cos\phi,$$

and $$\frac{y}{b}-\frac{x}{a}\sin\phi = \cos\phi.$$

Where these meet we have

$$\frac{x}{a}=\frac{y}{b},$$

i.e. they meet on an asymptote.

Hence, if PCP', dCd' are a pair of conjugate diameters of the hyperbola, the latter meeting its conjugate in d and d', the tangents at P, P', d, d' to the two hyperbolas form a parallelogram whose angular points are on their common asymptotes.

$$Pd^2 = a^2(\sec\phi-\tan\phi)^2+b^2(\tan\phi-\sec\phi)^2.$$

$$\therefore\ Pd = \sqrt{(a^2+b^2)}(\sec\phi-\tan\phi).$$

The equation of Pd is

$$\frac{x-a\sec\phi}{a(\sec\phi-\tan\phi)}=\frac{y-b\tan\phi}{b(\tan\phi-\sec\phi)},$$

i.e. $$bx+ay = ab(\sec\phi+\tan\phi).$$

The perpendicular from C $(0, 0)$ on this line is

$$\frac{ab(\sec\phi+\tan\phi)}{\sqrt{(a^2+b^2)}},$$

$\therefore$ the product of this perpendicular and Pd is

$$ab(\sec^2\phi-\tan^2\phi) = ab.$$

Hence the triangle CPd is of area $\frac{1}{2}ab$.

And the parallelogram formed by the tangents at P, P', d, d' is of area $4ab$, being 8 times the area of the triangle CPd.

Where Pd, whose equation is $bx+ay = ab(\sec\phi+\tan\phi)$, meets the asymptote $y=\frac{b}{a}x$, we have

$$x = \tfrac{1}{2}a(\sec\phi+\tan\phi), \qquad y = \tfrac{1}{2}b(\sec\phi+\tan\phi),$$

i.e. the point of intersection is the middle point of Pd.

It will be noted that the equations

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{of the hyperbola,}$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \quad \text{of its asymptotes,}$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1 \quad \text{of its conjugate hyperbola,}$$

only differ by a constant quantity.

6. The rectangular hyperbola

A *Rectangular Hyperbola* is defined as a hyperbola whose asymptotes are perpendicular to each other.

The asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ being $y=\pm\frac{b}{a}x$, if these are perpendicular, $\frac{b^2}{a^2}=1$.

The equation of a rectangular hyperbola referred to its axes is accordingly $\mathbf{x^2-y^2=a^2}$, and that of its conjugate

$$x^2-y^2=-a^2.$$

The gradients of the asymptotes are ± 1, i.e. they make angles of 45° with the axis and are therefore the bisectors of the angles between the axes.

In the relation $b^2=a^2(e^2-1)$, when $b^2=a^2$, $e=\sqrt{2}$. Thus the eccentricity of every rectangular hyperbola is $\sqrt{2}$.

From the fact that $b=a$, a rectangular hyperbola is sometimes called an *Equilateral Hyperbola*.

We shall now find the equation of a rectangular hyperbola referred to its asymptotes as axes of coordinates.

Turn the axes through an angle of $-45°$, keeping the origin unchanged, and in the equation

$$x^2-y^2=a^2$$

substitute for x, $x\cos 45°+y\sin 45°=\frac{x+y}{\sqrt{2}}$, and for y, $$-x\sin 45^\circ + y\cos 45^\circ = \frac{y-x}{\sqrt{2}}.$$

The equation then becomes

$$a^2 = \frac{(x+y)^2}{2} - \frac{(y-x)^2}{2} = 2xy,$$

which can be written $$\mathbf{xy = c^2},$$

where $$c^2 = \frac{a^2}{2}.$$

The conjugate hyperbola will clearly have its equation $xy = -c^2$ if we make the same substitutions for x and y in the equation $x^2-y^2 = -a^2$.

7. Parametric coordinates for $xy = c^2$

The coordinates of any point on the rectangular hyperbola $xy = c^2$ can be taken as $\left(c\mu, \frac{c}{\mu}\right)$, since these values of x and y satisfy the equation of the curve for any value of μ, the parameter.

When we speak of 'the point μ' on the rectangular hyperbola $xy = c^2$, we shall mean the point on it whose coordinates are $\left(c\mu, \frac{c}{\mu}\right)$ referred to its asymptotes as axes of coordinates.

8. Equations of the tangent and normal

The equation of the chord joining two points μ_1, μ_2 on the rectangular hyperbola $xy = c^2$ is

$$\frac{x-c\mu_1}{c(\mu_1-\mu_2)} = \frac{y-\dfrac{c}{\mu_1}}{c\left(\dfrac{1}{\mu_1}-\dfrac{1}{\mu_2}\right)} = \frac{\mu_2(\mu_1 y-c)}{c(\mu_2-\mu_1)},$$

i.e. $$\mathbf{x+\mu_1\mu_2 y = c(\mu_1+\mu_2)}.$$

The equation of the tangent at the point μ is obtained by putting $\mu_1 = \mu_2 = \mu$ in the equation of the chord just obtained and is

$$\mathbf{x+\mu^2 y = 2c\mu}.$$

The coordinates of the vertex A at the extremity of the transverse axis of the rectangular hyperbola $xy = c^2$ are clearly $a\cos 45^\circ$, $a\sin 45^\circ$, i.e. (c, c), and the parameter of A is 1.

The tangents at the vertices A, A' of the transverse axis are therefore

$$x+y = \pm 2c,$$

which, we note, are perpendicular to the line $y = x$, the axis AA'.

The equation of the chord joining two points (x_1, y_1), (x_2, y_2) on the hyperbola $xy = c^2$ is

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} = \frac{y-y_1}{c^2\left(\dfrac{1}{x_1}-\dfrac{1}{x_2}\right)},$$

i.e.

$$\begin{aligned} c^2x+x_1x_2y &= c^2x_1+x_1y_1x_2 \\ &= c^2(x_1+x_2). \end{aligned}$$

Putting $x_2 = x_1$ in this equation, we get the equation of the tangent at (x_1, y_1) to be

$$\frac{x}{x_1}+\frac{x_1 y}{c^2} = 2,$$

i.e.

$$\frac{x}{x_1}+\frac{y}{y_1} = 2,$$

a result which could also be obtained either (1) by writing the parametric equation $\dfrac{x}{c\mu}+\dfrac{y}{\dfrac{c}{\mu}} = 2$ and putting $x_1 = c\mu$, $y_1 = \dfrac{c}{\mu}$ in it, or (2) by differentiating $xy = c^2$ and finding that $\dfrac{dy}{dx} = -\dfrac{y}{x}$, whence the equation of the tangent at (x_1, y_1) is

$$y-y_1 = -\frac{y_1}{x_1}(x-x_1),$$

which reduces to $$\frac{\mathbf{x}}{\mathbf{x}_1}+\frac{\mathbf{y}}{\mathbf{y}_1}=\mathbf{2}.$$

Where the tangent at a point μ cuts the asymptotes $x=0$, $y=0$, we have, from the equation $x+y\mu^2=2c\mu$,

$$x=2c\mu, \qquad y=\frac{2c}{\mu}.$$

The middle point of the line joining $(2c\mu, 0)$, $\left(0, \frac{2c}{\mu}\right)$ is clearly the point $\left(c\mu, \frac{c}{\mu}\right)$, i.e. the point at which the tangent touches the hyperbola.

Hence the portion of any tangent intercepted between the asymptotes is bisected at the point of contact.

The area of the triangle formed by any tangent and the asymptotes is also clearly $\frac{1}{2}.2c\mu\frac{2c}{\mu}=2c^2$.

The normal at the point μ of the rectangular hyperbola $xy=c^2$ is the line through $\left(c\mu, \frac{c}{\mu}\right)$ perpendicular to

$$x+y\mu^2=2c\mu.$$

Hence its equation is

$$\mu^2x-y=\mu^2.c\mu-\frac{c}{\mu},$$

i.e. $$\mathbf{c\mu^4-\mu^3x+\mu y-c=0.}$$

In non-parametric coordinates the normal at (x', y') perpendicular to $\frac{x}{x'}+\frac{y}{y'}=2$ at the point (x', y') is

$$x'(x-x')-y'(y-y')=0,$$

i.e. $$\mathbf{xx'-yy'=x'^2-y'^2.}$$

If the tangent at the point μ, viz. $x+y\mu^2=2c\mu$, passes through the point (h, k), we have

$$\mu^2k-2c\mu+h=0.$$

As this is a quadratic in μ, it indicates that there will be two values for μ, i.e. two points of contact, and therefore two tangents can be drawn to the curve from any point (h, k) which will be real, coincident, or imaginary, according as $c^2 \gtreqless hk$, i.e. according as the point is outside, upon, or within the curve.

Again, as the equation of the normal is

$$c\mu^4 - \mu^3 x + \mu y - c = 0,$$

if any point (h, k) lies on it,

$$c\mu^4 - \mu^3 h + \mu k - c = 0,$$

giving four values of μ, thus indicating that in general four normals can be drawn to the rectangular hyperbola from any given point (h, k).

9. Tangential equation of a hyperbola

To find the tangential equation of $xy = c^2$ we have to find the condition that the line $lx + my + n = 0$ should touch it.

Identifying $lx + my + n = 0$ with $x + y\mu^2 - 2c\mu = 0$, we have

$$\frac{1}{l} = \frac{\mu^2}{m} = -\frac{2c\mu}{n},$$

whence $$\frac{m}{l} = \mu^2 = \frac{n^2}{4c^2 l^2}$$

and $$\mathbf{n^2 = 4c^2 lm},$$

the required tangential equation.

10. Conjugate diameters

As previously stated, the condition that $y = mx$, $y = m'x$ should be conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is that $mm' = \frac{b^2}{a^2}$.

In the case of the rectangular hyperbola, where $b^2 = a^2$, it follows that $y - mx = 0$ and $my - x = 0$ will always be conjugate.

Transferring to the asymptotes as axes of coordinates,

$$\frac{x-y}{\sqrt{2}}-m\frac{x+y}{\sqrt{2}}=0, \quad \text{i.e. } x(1-m)=y(1+m),$$

and $$\frac{m(x-y)}{\sqrt{2}}-\frac{x+y}{\sqrt{2}}=0, \quad \text{i.e. } x(m-1)=y(1+m),$$

will be conjugate diameters, and their gradients being equal and of opposite sign, it follows that they are equally inclined to the asymptotes.

11. The polar

The tangents at the points μ_1, μ_2 of the rectangular hyperbola $xy=c^2$ being $x+y\mu_1^2=2c\mu_1$ and $x+y\mu_2^2=2c\mu_2$ respectively, where they intersect we have

$$\mathbf{x}=\frac{2c\mu_1\mu_2}{\mu_1+\mu_2} \quad \text{and} \quad \mathbf{y}=\frac{2c}{\mu_1+\mu_2}.$$

Suppose that the tangents at μ_1 and μ_2 meet at a point (h,k).

Then $$\mu_1+\mu_2=\frac{2c}{k} \quad \text{and} \quad \mu_1\mu_2=\frac{h}{k}.$$

The equation of the line joining the points μ_1, μ_2, which is the polar of (h,k), is

$$x+y\mu_1\mu_2=c(\mu_1+\mu_2),$$

i.e. $$x+\frac{h}{k}y=\frac{2c^2}{k},$$

i.e. $$\frac{x}{h}+\frac{y}{k}=\frac{2c^2}{hk}.$$

Note that this is not quite the same form as the equation of the tangent as has been the case previously.

12. Equation of two tangents from a point

Let (h,k) be the point from which tangents are drawn to the rectangular hyperbola $xy=c^2$.

The coordinates of any point being (X, Y), those of the point dividing (h, k) and (X, Y) in the ratio $p : q$ are

$$\frac{pX+qh}{p+q}, \frac{pY+qk}{p+q}.$$

Substituting these in the equation $xy = c^2$, we have

$$\frac{(pX+qh)(pY+qk)}{(p+q)^2} = c^2,$$

i.e. $$\frac{p^2}{q^2}(XY-c^2)+\frac{p}{q}(Xk+Yh-2c^2)+hk-c^2 = 0.$$

Now if the line joining (h, k) to (X, Y) is a tangent to $xy = c^2$, this equation in $\frac{p}{q}$ must have equal roots. The condition for this is

$$(Xk+Yh-2c^2)^2 = 4(XY-c^2)(hk-c^2),$$

which is the equation of the tangents to the rectangular hyperbola $xy = c^2$ from the point (h, k).

13. Some characteristic properties of a rectangular hyperbola

(1) Any chord of a rectangular hyperbola subtends equal or supplementary angles at the extremities of any diameter.

Let μ_1, μ_2 be the extremities of a chord of the rectangular hyperbola $xy = c^2$, and λ, $-\lambda$ the extremities of any diameter.

The gradient of the line joining λ, μ_1 is $-\frac{1}{\lambda\mu_1}$.

That of the line joining λ, μ_2 is $-\frac{1}{\lambda\mu_2}$.

The tangent of the angle between these lines is

$$\frac{-\frac{1}{\lambda\mu_1}+\frac{1}{\lambda\mu_2}}{1+\frac{1}{\lambda^2\mu_1\mu_2}} = \frac{\lambda(\mu_1-\mu_2)}{1+\lambda^2\mu_1\mu_2}.$$

Similarly, the tangent of the angle between the lines joining $-\lambda$, μ_1, and $-\lambda$, μ_2, is

$$-\frac{\lambda(\mu_1-\mu_2)}{1+\lambda^2\mu_1\mu_2}.$$

$\therefore$ the angles are supplementary or equal.

(2) If a rectangular hyperbola circumscribes a triangle it must pass through the orthocentre.

Take three points μ_1, μ_2, μ_3 on the rectangular hyperbola $xy = c^2$.

The chord joining the points μ_2, μ_3 is $x+y\mu_2\mu_3 = c(\mu_2+\mu_3)$.

The line perpendicular to this chord through the point μ_1 is

$$\mu_2\mu_3x-y = c\mu_1\mu_2\mu_3-\frac{c}{\mu_1}.$$

Similarly, the line through μ_2 perpendicular to the chord joining the points μ_1, μ_3 is

$$\mu_1\mu_3x-y = c\mu_1\mu_2\mu_3-\frac{c}{\mu_2}.$$

Where these two lines intersect, the orthocentre of the triangle, we have

$$\mu_3(\mu_1-\mu_2)x = \frac{c}{\mu_1}-\frac{c}{\mu_2} = -\frac{c(\mu_1-\mu_2)}{\mu_1\mu_2},$$

i.e.
$$x = -\frac{c}{\mu_1\mu_2\mu_3},$$

and
$$y = \frac{c}{\mu_1}-c\mu_1\mu_2\mu_3+\mu_2\mu_3x = -c\mu_1\mu_2\mu_3.$$

These coordinates of the orthocentre $\left(-\frac{c}{\mu_1\mu_2\mu_3}, -c\mu_1\mu_2\mu_3\right)$ clearly satisfy the equation $xy = c^2$.

$\therefore$ the orthocentre of the triangle is on the hyperbola.

(3) The normal to the rectangular hyperbola at the point μ cuts the curve again in the point $-\frac{1}{\mu^3}$.

Identify the chord joining the two points μ, μ' on the rectangular hyperbola $xy = c^2$,

$$x+y\mu\mu' = c(\mu+\mu'),$$

with the normal at μ, viz.

$$\mu^2x-y = c\mu^3-\frac{c}{\mu},$$

and we have $\mu^2 = -\frac{1}{\mu\mu'}$, i.e. $\mu' = -\frac{1}{\mu^3}$.

(4) If *PQR be a right-angled triangle inscribed in a rectangular hyperbola, the angle at P being right, then the tangent at P is perpendicular to QR.*

Let P, Q, R be the three points μ_1, μ_2, μ_3.

The equation of PQ is

$$x+y\mu_1\mu_2 = c(\mu_1+\mu_2)$$

and that of PR is $x+y\mu_1\mu_3 = c(\mu_2+\mu_3)$.

These being perpendicular, we have

$$\frac{1}{\mu_1^2\mu_2\mu_3} = -1.$$

But the equation of the chord QR is $x+y\mu_2\mu_3 = c(\mu_2+\mu_3)$; and that of the tangent at P, $x+y\mu_1^2 = 2c\mu_1$.

Thus the product of the gradients of QR and the tangent at P is

$$\frac{1}{\mu_1^2\mu_2\mu_3} = -1.$$

$\therefore$ the tangent at P is perpendicular to the chord QR.

(5) *If P be a point on a rectangular hyperbola, then* $SP.S'P = CP^2$, *C being its centre.*

Let P be the point $(a\sec\phi, a\tan\phi)$ on the rectangular hyperbola $x^2-y^2 = a^2$.

Then

$$SP = e\left(a\sec\phi-\frac{a}{e}\right) = a(e\sec\phi-1),$$

$$S'P = e\left(a\sec\phi+\frac{a}{e}\right) = a(e\sec\phi+1).$$

Now $e = \sqrt{2}$.

$$\therefore\ SP.S'P = a^2(e^2\sec^2\phi - 1) = a^2(2\sec^2\phi - 1).$$

Again

$$CP^2 = a^2\sec^2\phi + a^2\tan^2\phi = a^2(2\sec^2\phi - 1).$$

$$\therefore\ SP.S'P = CP^2.$$

(6) *If the normal to a rectangular hyperbola at any point cuts the transverse axis in G and the conjugate axis in g, Gg is bisected at the point.*

Let μ be any point on the rectangular hyperbola $xy = c^2$.

The equation of the normal at μ is

$$\mu^2x - y = c\mu^3 - \frac{c}{\mu}.$$

Where this cuts $y = x$ we have

$$y = x = \frac{c(\mu^4-1)}{\mu(\mu^2-1)} = \frac{c}{\mu}(\mu^2+1) = c\mu + \frac{c}{\mu}.$$

Where it cuts $y = -x$ we have

$$-y = x = \frac{c(\mu^4-1)}{\mu(\mu^2+1)} = \frac{c}{\mu}(\mu^2-1) = c\mu - \frac{c}{\mu}.$$

The middle point of the line joining these two points is clearly $\left(c\mu, \frac{c}{\mu}\right)$, i.e. the point μ.

14. Conormal points on a rectangular hyperbola

The equation of the normal at a point μ on the rectangular hyperbola $xy = c^2$ is

$$c\mu^4 - \mu^3x + \mu y - c = 0.$$

If this passes through the point (h, k),

$$c\mu^4 - \mu^3h + \mu k - c = 0.$$

Let μ_1, μ_2, μ_3, μ_4 be the roots of this equation.

Then $\mu_1\mu_2\mu_3\mu_4 = -1$. $\therefore\ c^4\mu_1\mu_2\mu_3\mu_4 = \dfrac{c^4}{\mu_1\mu_2\mu_3\mu_4} = -c^4$; which proves that the product of the abscissae of four conormal points is equal in magnitude to the product of their ordinates.

We see also that

$$\frac{h}{c} = \mu_1+\mu_2+\mu_3+\mu_4 \quad\text{and}\quad -\frac{k}{c} = \mu_1\mu_2\mu_3\mu_4\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}+\frac{1}{\mu_4}\right),$$

i.e. the four normals meet at a point whose coordinates are

$$c(\mu_1+\mu_2+\mu_3+\mu_4) \quad\text{and}\quad c\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}+\frac{1}{\mu_4}\right).$$

As there is no coefficient of μ^2 in the equation considered,

$$\therefore\ \mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1+\mu_4(\mu_1+\mu_2+\mu_3) = 0.$$

15. Concyclic points on a rectangular hyperbola

Let the equation of the circle be $x^2+y^2+2gx+2fy+d = 0$.

The points where it cuts the rectangular hyperbola $xy = c^2$ are given by the equation

$$c^2\mu^2+\frac{c^2}{\mu^2}+2gc\mu+2f\frac{c}{\mu}+d = 0,$$

i.e.
$$c^2\mu^4+2gc\mu^3+d\mu^2+2fc\mu+c^2 = 0.$$

If μ_1, μ_2, μ_3, μ_4 are the roots of this equation,

$$\mu_1\mu_2\mu_3\mu_4 = 1,$$

so that the product of the abscissae or of the ordinates of four concyclic points on a rectangular hyperbola $xy = c^2$ is c^4.

The gradient of the chord joining two points, say μ_1 and μ_2, is $-\dfrac{1}{\mu_1\mu_2}$.

The gradient of the chord joining the other two points is $-\dfrac{1}{\mu_3\mu_4} = -\mu_1\mu_2$, since $\mu_1\mu_2\mu_3\mu_4 = 1$.

So if the chords make angles of θ_1, θ_2 with the asymptote which is the x-axis

$$\tan\theta_1 = \cot\theta_2 = \tan\left(\frac{\pi}{2}-\theta_2\right) \quad \text{or} \quad \tan\left(\frac{3\pi}{2}-\theta_2\right).$$

$$\therefore\ \theta_1+\theta_2 = \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2}.$$

The angle the chord joining μ_1, μ_2 makes with the line $y = x$, the transverse axis, is

$$\tan^{-1}\frac{1+\dfrac{1}{\mu_1\mu_2}}{1-\dfrac{1}{\mu_1\mu_2}}, \quad \text{i.e. } \tan^{-1}\frac{\mu_1\mu_2+1}{\mu_1\mu_2-1}.$$

The angle the chord joining μ_3, μ_4 makes with it is, similarly,

$$\tan^{-1}\frac{\mu_3\mu_4+1}{\mu_3\mu_4-1} = \tan^{-1}\frac{1+\mu_1\mu_2}{1-\mu_1\mu_2},$$

since $\mu_1\mu_2\mu_3\mu_4 = 1$.

The chords are therefore equally inclined to the transverse axis of the rectangular hyperbola.

16. Typical examples

Ex. 1. Find the lengths of the axes, the eccentricity, the coordinates of the focus, the latus rectum, and the equations of the axes, directrices, and asymptotes of the hyperbola whose equation is $16x^2-81y^2-48x-108y-4 = 0$.

The equations giving the centre are

$$32x-48 = 0, \qquad -162y-108 = 0,$$

whence $x = \frac{3}{2}$, $y = -\frac{2}{3}$.

Transferring the equation to parallel axes through this point, we have

$$16(x+\tfrac{3}{2})^2-81(y-\tfrac{2}{3})^2-48(x+\tfrac{3}{2})-108(y-\tfrac{2}{3})-4 = 0,$$

which reduces to $$\frac{x^2}{(\frac{1}{2})^2}-\frac{y^2}{(\frac{2}{9})^2} = 1.$$

The lengths of the axes are clearly 1 and $\frac{4}{9}\sqrt{(-1)}$.

The eccentricity is given by the equation $\frac{4}{81} = \frac{1}{4}(e^2-1)$, whence $e = \frac{\sqrt{97}}{9}$.

The coordinates of the focus are $\pm\frac{\sqrt{97}}{18}, 0$, or referred to the original axes, $\frac{3}{2}\pm\frac{\sqrt{97}}{18}, -\frac{2}{3}$.

The length of the latus rectum $= 2a(e^2-1) = \frac{16}{81}$.

The equations of the axes, directrices, and asymptotes are (1) $x = 0$, $y = 0$; (2) $x = \pm\frac{9}{2\sqrt{97}}$; (3) $16x^2-81y^2 = 0$, referred to the new axes, and

(1) $2x-3 = 0$, $3y+2 = 0$;

(2) $x = \frac{3}{2}\pm\frac{9}{2\sqrt{97}}$;

(3) $16(x-\frac{3}{2})^2-81(y+\frac{2}{3})^2 = 0$;

which is
$$16x^2-81y^2-48x-108y = 0,$$
referred to the original ones.

Ex. 2. Show that the chord of a hyperbola which subtends a right angle at a vertex will always cut the transverse axis in a fixed point.

The equation of a hyperbola referred to its transverse axis and the tangent at the vertex as axes of coordinates can be written
$$\frac{(x+a)^2}{a^2}-\frac{y^2}{b^2} = 1.$$

The lines joining the vertex (0, 0) to where the line $lx+my = 1$ cuts this hyperbola have as their equation
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{2x}{a}(lx+my) = 0.$$

If these are perpendicular, then
$$\frac{1}{a^2}+\frac{2l}{a}-\frac{1}{b^2} = 0.$$

$$\therefore\ l = \frac{a}{2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right),\quad \text{a constant.}$$

Hence the point $\left(\frac{1}{l}, 0\right)$ in which the line $lx+my = 1$ cuts the transverse axis is in this case a fixed one.

Ex. 3. Find the locus of the middle points of chords of a rectangular hyperbola which pass through a fixed point (h, k).

Let (X, Y) be the middle point of the chord joining two points μ_1, μ_2 on the rectangular hyperbola $xy = c^2$.

Then $2X = c(\mu_1+\mu_2)$, $2Y = c\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)$, and $\frac{X}{Y} = \mu_1\mu_2$.

The point (h, k) lies on the chord $x+\mu_1\mu_2 y = c(\mu_1+\mu_2)$.

$$\therefore\ h+\mu_1\mu_2 k = c(\mu_1+\mu_2)$$

and
$$h+k\frac{X}{Y} = 2X,$$

i.e.
$$2XY = kX+hY$$

is the required locus.

Writing it
$$\left(X-\frac{h}{2}\right)\left(Y-\frac{k}{2}\right) = \frac{hk}{4},$$

we see that it is also a rectangular hyperbola with the point $\left(\frac{h}{2}, \frac{k}{2}\right)$ as centre and asymptotes parallel to those of $xy = c^2$.

Ex. 4. *TP*, *TQ* are two tangents to a rectangular hyperbola. Show that the bisectors of the angles *PTQ* and *PCQ* meet on the chord of contact *PQ*.

Let *P*, *Q* be the points μ_1, μ_2 on the rectangular hyperbola $xy = c^2$.

The tangents at *P* and *Q* are

$$x+y\mu_1^2 = 2c\mu_1 \quad\text{and}\quad x+y\mu_2^2 = 2c\mu_2.$$

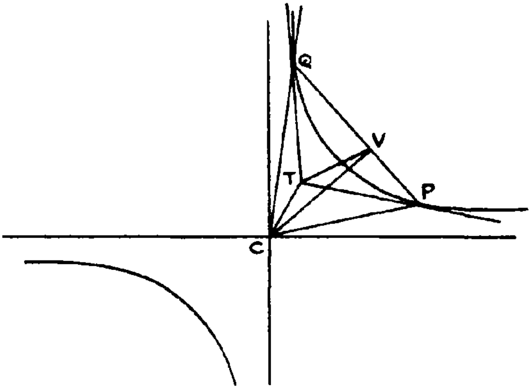
At *T*, where they meet,

$$x = \frac{2c\mu_1\mu_2}{\mu_1+\mu_2}, \qquad y = \frac{2c}{\mu_1+\mu_2}.$$

$$\therefore\ TP^2 = \left(\frac{2c\mu_1\mu_2}{\mu_1+\mu_2} - c\mu_1\right)^2 + \left(\frac{2c}{\mu_1+\mu_2} - \frac{c}{\mu_1}\right)^2$$

$$= \frac{c^2\mu_1^2(\mu_1-\mu_2)^2}{(\mu_1+\mu_2)^2} + \frac{c^2(\mu_1-\mu_2)^2}{\mu_1^2(\mu_1+\mu_2)^2} = \frac{(\mu_1-\mu_2)^2}{(\mu_1+\mu_2)^2}\left(c^2\mu_1^2 + \frac{c^2}{\mu_1^2}\right)$$

$$= \frac{(\mu_1-\mu_2)^2}{(\mu_1+\mu_2)^2}\,.\,CP^2.$$



Similarly, $$TQ^2 = \frac{(\mu_1-\mu_2)^2}{(\mu_1+\mu_2)^2}\,.\,CQ^2.$$

$$\therefore\ \frac{TP}{TQ} = \frac{CP}{CQ}.$$

Suppose that the bisector of $\angle PTQ$ cuts PQ in V.

Then $$\frac{CP}{CQ} = \frac{TP}{TQ} = \frac{PV}{QV}.$$

$\therefore$ CV is the bisector of $\angle PCQ$.

Ex. 5. If four points on a rectangular hyperbola are such that the chord joining one pair is perpendicular to the chord joining another pair, show that the same must be the case with the other pairs of chords joining the points.

Let μ_1, μ_2, μ_3, μ_4 be four points on the rectangular hyperbola $xy = c^2$.

Then we are given that $x+y\mu_1\mu_2 = c(\mu_1+\mu_2)$, say, is perpendicular to $x+y\mu_3\mu_4 = c(\mu_3+\mu_4)$.

$$\therefore\ \frac{1}{\mu_1\mu_2\mu_3\mu_4} = -1,$$

and this condition being satisfied it is clear that

$$x+y\mu_1\mu_3 = c(\mu_1+\mu_3) \quad\text{and}\quad x+y\mu_2\mu_4 = c(\mu_2+\mu_4)$$

or $$x+y\mu_1\mu_4 = c(\mu_1+\mu_4) \quad\text{and}\quad x+y\mu_2\mu_3 = c(\mu_2+\mu_3)$$

are also pairs of chords perpendicular to each other.

Ex. 6. A line drawn through a fixed point O cuts a rectangular hyperbola in the points P and Q and the polar of O in R. Prove that $\frac{1}{OP}+\frac{1}{OQ} = \frac{2}{OR}$.

Let the equation of the rectangular hyperbola be $xy = c^2$ and let O be the point (h, k).

Where the line $\frac{x-h}{\cos\theta} = r = \frac{y-k}{\sin\theta}$ meets the hyperbola, we have

$$(h+r\cos\theta)(k+r\sin\theta) = c^2,$$

i.e. $$r^2\sin\theta\cos\theta + r(h\sin\theta + k\cos\theta) + hk - c^2 = 0.$$

If the roots of this equation are r_1, r_2, then

$$\frac{1}{OP}+\frac{1}{OQ} = \frac{1}{r_1}+\frac{1}{r_2} = -\frac{h\sin\theta + k\cos\theta}{hk-c^2}.$$

The equation of the polar of O is

$$\frac{x}{h}+\frac{y}{k} = \frac{2c^2}{hk}.$$

Where the line OPQ cuts this, we have

$$\frac{h+r\cos\theta}{h}+\frac{k+r\sin\theta}{k} = \frac{2c^2}{hk}.$$

$$\therefore \ OR = r = \frac{2(c^2-hk)}{h\sin\theta + k\cos\theta}.$$

$$\therefore \ \frac{2}{OR} = \frac{h\sin\theta + k\cos\theta}{c^2-hk} = \frac{1}{OP} + \frac{1}{OQ}.$$

Ex. 7. Find the coordinates of the point in which the normal at a point μ of the rectangular hyperbola $xy = c^2$ again meets the curve.

The normals at two points P, Q on the curve meet it again at R and S respectively. Show that, if the line PQ is fixed in direction, the line RS is also fixed in direction. [L.]

Let P and R be the points λ, λ', and Q and S the points μ, μ' on the rectangular hyperbola $xy = c^2$.

The equation of PR is $x + \lambda\lambda' y = c(\lambda+\lambda')$, and of the normal at P, $\lambda^2 x - y = c\lambda^3 - \dfrac{c}{\lambda}$.

If these are identical, we have $\lambda\lambda' = -\dfrac{1}{\lambda^2}$, i.e. $\lambda' = -\dfrac{1}{\lambda^3}$.

Hence the coordinates of the point in which the normal at P cuts the curve again are $-\dfrac{c}{\lambda^3}$, $-c\lambda^3$.

The equation of PQ is $x + \lambda\mu y = c(\lambda+\mu)$. If this is fixed in direction, then $\lambda\mu$ is a constant.

Again, the equation of RS, the line joining the points $-\dfrac{1}{\lambda^3}$, $-\dfrac{1}{\mu^3}$, is

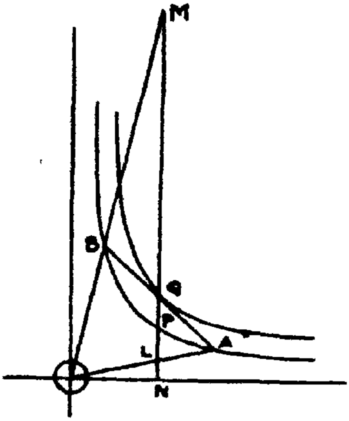
$$x + \frac{1}{\lambda^3\mu^3} y = -c\left(\frac{1}{\lambda^3} + \frac{1}{\mu^3}\right).$$

But as $\lambda\mu$ is constant, $\therefore$ this line is also fixed in direction.

Ex. 8. Two rectangular hyperbolas have the same centre O and the same asymptotes. A line NPQ is drawn parallel to one asymptote to meet the other asymptote at N and the hyperbolas at P, Q. The tangent at Q to the second hyperbola meets the first at A and B, and OA, OB cut NPQ at L and M. Find the equation of the pair of lines OA, OB and hence show that $NL.NM = NQ^2$. [J.]

Let the two hyperbolas be $xy = c^2$, $xy = c'^2$, and let NPQ be the line $x = c\mu$ parallel to the asymptote $y = 0$.

Then P is the point $\left(c\mu, \frac{c}{\mu}\right)$ and Q the point $\left(c\mu, \frac{c'^2}{c\mu}\right)$.



The equation of the tangent at the point Q whose parameter is $\frac{c}{c'}\mu$ is

$$x + y\frac{c^2\mu^2}{c'^2} = 2c'.\frac{c}{c'}\mu,$$

i.e.
$$\frac{x}{2c\mu} + \frac{yc\mu}{2c'^2} = 1.$$

The lines joining O to the points where $xy = c^2$ is cut by this line are

$$xy = \frac{c^2}{4}\left(\frac{x}{c\mu} + \frac{yc\mu}{c'^2}\right)^2.$$

The ordinates of the points L and M in which these lines OA and OB are cut by the line $x = c\mu$ are the roots of the equation

$$c\mu y = \frac{c^2}{4}\left(1 + \frac{yc\mu}{c'^2}\right)^2.$$

If y_1 and y_2 are the roots of this equation,

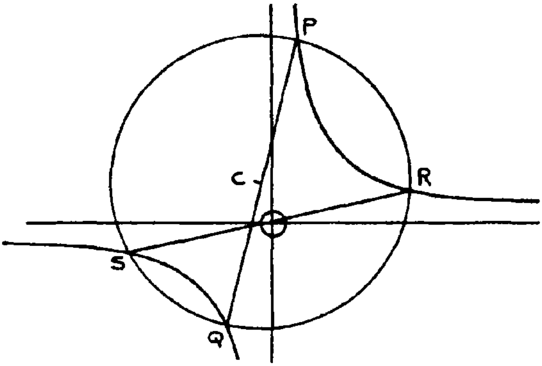
$$NL.NM = y_1 y_2 = \left(\frac{c'^2}{c\mu}\right)^2 = NQ^2.$$

Ex. 9. A circle cuts a rectangular hyperbola in the points P, Q, R, S. Prove that, if the middle point of the line joining a pair of these points is the centre of the circle, the middle point of the line joining the remaining pair of points is the centre of the hyperbola.

Let P, Q, R, S be the four points $\mu_1, \mu_2, \mu_3, \mu_4$ on the hyperbola $xy = c^2$, where it is intersected by the circle whose equation is $x^2+y^2+2gx+2fy+d = 0$.

Then, as in Art. 15, $\mu_1, \mu_2, \mu_3, \mu_4$ are the roots of the equation

$$c^2\mu^4+2gc\mu^3+d\mu^2+2fc\mu+c^2 = 0.$$



$$\therefore\ \mu_1+\mu_2+\mu_3+\mu_4 = -\frac{2g}{c}$$

and

$$\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}+\frac{1}{\mu_4} = \frac{\mu_1\mu_2\mu_3+\mu_1\mu_2\mu_4+\mu_1\mu_3\mu_4+\mu_2\mu_3\mu_4}{\mu_1\mu_2\mu_3\mu_4}$$

$$= -\frac{2f}{c}.$$

So if $\quad \frac{c}{2}(\mu_1+\mu_2) = -g \quad$ and $\quad \frac{c}{2}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right) = -f,$

we have $\quad \frac{c}{2}(\mu_3+\mu_4) = 0 \quad$ and $\quad \frac{c}{2}\left(\frac{1}{\mu_3}+\frac{1}{\mu_4}\right) = 0,$

i.e. the middle point of the line joining μ_3, μ_4 is the centre of the rectangular hyperbola.

Ex. 10. The circles described on parallel chords of a rectangular hyperbola as diameters have a common radical axis.

Let μ_1, μ_2 be the extremities of one chord of the parallel system with regard to the rectangular hyperbola $xy = c^2$.

The equation of the circle on the line joining the points μ_1, μ_2 as diameter is

$$(x-c\mu_1)(x-c\mu_2)+\left(y-\frac{c}{\mu_1}\right)\left(y-\frac{c}{\mu_2}\right)=0,$$

i.e. $$x^2+y^2-cx(\mu_1+\mu_2)-cy\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)+c^2\mu_1\mu_2+\frac{c^2}{\mu_1\mu_2}=0.$$

Similarly, the equation of the circle on the line joining the points μ_3, μ_4 as diameter is

$$x^2+y^2-cx(\mu_3+\mu_4)-cy\left(\frac{1}{\mu_3}+\frac{1}{\mu_4}\right)+c^2\mu_3\mu_4+\frac{c^2}{\mu_3\mu_4}=0.$$

But as the chords joining μ_1, μ_2 and μ_3, μ_4 are parallel,

$$\therefore\ \mu_3\mu_4=\mu_1\mu_2.$$

The radical axis of the two circles, obtained by subtraction, is

$$cx(\mu_1+\mu_2-\mu_3-\mu_4)+cy\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}-\frac{1}{\mu_3}-\frac{1}{\mu_4}\right)=0,$$

i.e. $$cx(\mu_1+\mu_2-\mu_3-\mu_4)+cy\left(\frac{\mu_1+\mu_2}{\mu_1\mu_2}-\frac{\mu_3+\mu_4}{\mu_3\mu_4}\right)=0,$$

and since $\mu_1\mu_2=\mu_3\mu_4$, this equation becomes

$$kx+y=0,$$

where $\mu_1\mu_2=\mu_3\mu_4=$ a constant k.

$\therefore$ all the circles are coaxal and their radical axis is the line $my=x$, where m is the the gradient of any chord of the parallel system.

Ex. 11. The equation of an ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Rectangular hyperbolas are drawn with the axes of this ellipse as their asymptotes, the equation of any one of them being $xy=c^2$. Show that the common chords of the ellipse and the hyperbolas are parallel to one of the equi-conjugate diameters of the ellipse.

Substitute $x = c\mu$, $y = \frac{c}{\mu}$ in the equation of the ellipse and we have

$$\frac{c^2\mu^2}{a^2}+\frac{c^2}{b^2\mu^2} = 1, \quad \text{i.e. } \frac{\mu^4}{a^2}-\frac{\mu^2}{c^2}+\frac{1}{b^2} = 0.$$

Suppose that the roots of this equation are $\pm\mu_1$, $\pm\mu_2$.

Then $\qquad \mu_1^2\mu_2^2 = \frac{a^2}{b^2} \quad$ and $\quad \mu_1\mu_2 = \pm\frac{a}{b}$.

The gradient of the chord joining μ_1, μ_2, say, is $-\frac{1}{\mu_1\mu_2} = \mp\frac{b}{a}$.

Hence the chord is parallel to either $y = -\frac{b}{a}x$ or $y = \frac{b}{a}x$, i.e. to one of the equi-conjugate diameters of the ellipse.

Ex. 12. Show that the asymptotes of the conic whose equation is

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

are always parallel to the lines $ax^2+2hxy+by^2 = 0$ and that the general equation will represent a hyperbola, a parabola, or an ellipse according as $h^2 \gtreqless ab$.

As the equation of a conic only differs from that of its asymptotes by a fixed quantity, λ say, the equation of the asymptotes of the conic are

$$ax^2+2hxy+by^2+2gx+2fy+c+\lambda = 0,$$

and as these have to be straight lines,

$$\therefore\ ab(c+\lambda)+2fgh-af^2-bg^2-(c+\lambda)h^2 = 0.$$

$$\therefore\ \lambda = \frac{abc+2fgh-af^2-bg^2-ch^2}{h^2-ab} = \frac{\Delta}{h^2-ab},$$

where Δ is the discriminant.

The equation of the asymptotes of the conic is therefore

$$ax^2+2hxy+by^2+2gx+2fy+c-\frac{\Delta}{ab-h^2} = 0,$$

and they are clearly parallel to the lines

$$ax^2+2hxy+by^2 = 0.$$

If the last equation represents two real lines, then $h^2 > ab$ and the conic is a hyperbola. If $a+b = 0$, it is a rectangular hyperbola.

If $h^2 < ab$, the lines are imaginary and the conic is an ellipse.

If $h^2 = ab$, the terms of the second degree in the equation are a perfect square and the conic is a parabola.

EXAMPLES 14

1. Find the ordinates of the points on the hyperbola

$$4x^2 - 9y^2 = 36$$

whose abscissae are 6, -9, and 12 respectively.

2. Find the value of the parameter in the following cases:

(1) the point $(-10, -3\sqrt{3})$ on the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$;

(2) the point $(6\sqrt{2}, -6)$ on the rectangular hyperbola

$$x^2 - y^2 = 36;$$

(3) the point $(16, 9)$ on the rectangular hyperbola $xy = 144$.

3. The angle between the asymptotes of a hyperbola is 60°. Find its eccentricity.

4. Find the eccentricity of a hyperbola whose latus rectum is double the transverse axis.

5. Find the coordinates of the vertices and foci, also the equation of the directrix of the rectangular hyperbola $xy = c^2$.

6. Find the coordinates of the centre and foci, the eccentricity, and the lengths of the transverse and conjugate axes of the following hyperbolas:

(1) $25x^2 - 49y^2 = 35^2$.
(2) $9(x-2)^2 - 64(y+3)^2 = 576$.
(3) $25x^2 - 9y^2 - 20x - 12y - 225 = 0$.

7. Find the equation of the asymptotes of each of the hyperbolas in the previous question.

8. Show that one of the asymptotes of the rectangular hyperbola $x(\alpha y + \beta) = c^2$ passes through the origin.

9. Find the equations of the following hyperbolas:

(1) Focus at the origin, directrix the line $3x-4y = 1$, and the eccentricity $\frac{5}{4}$.

(2) Focus the point $(-1, 2)$, directrix the line $5x-12y = 4$, and the eccentricity $\frac{3}{2}$.

(3) Centre at the origin, directrix the line $3x-4y = 2$, and the eccentricity 2.

10. Find the equation of the tangents and normals to the following hyperbolas:

(1) $x^2-y^2 = 49$ at the point $(8, \sqrt{15})$.

(2) $49x^2-64y^2 = 56^2$ at the point $(16, 7\sqrt{3})$.

(3) $36x^2-49y^2+216x-196y-1636 = 0$ at the point $(11, 6\sqrt{3}-2)$.

11. Find the equation of the normal to the hyperbola $\frac{x^2}{25}-\frac{y^2}{9} = 1$ at the point $(10, 3\sqrt{3})$.

12. Find the coordinates of the point in which the line

$$4x+\sqrt{3}y = 1$$

touches the hyperbola $4x^2-y^2 = 1$.

13. Show that the line $y = x-2\sqrt{3}$ is a tangent to the hyperbola $x^2-4y^2 = 16$.

14. Show that the line $2bx-ay = \sqrt{3}ab$ is a tangent to the hyperbola $b^2x^2-a^2y^2 = a^2b^2$, and find the coordinates of its point of contact.

15. If the line $2\sqrt{2}x-3y-c = 0$ touches the hyperbola $\frac{x^2}{36}-\frac{y^2}{16} = 1$, find the value of c.

16. Find the point at which the line $7\sqrt{3}x+4\sqrt{3}y = 106$ is a normal to the hyperbola $4x^2-49y^2 = 196$.

17. The line $y = 4x$ is a diameter of the hyperbola $\frac{x^2}{64}-\frac{y^2}{36} = 1$. What is the equation of its conjugate diameter?

18. If the tangent at any point P of the hyperbola

$$b^2x^2-a^2y^2 = a^2b^2$$

meet the tangents at the vertices A, A' in K and K' respectively, show that $AK . A'K' = b^2$.

19. Show that a hyperbola cannot have equi-conjugate diameters.

20. Prove that the area of the triangle between a tangent and the asymptotes of a hyperbola is constant.

21. If the eccentricities of two conjugate hyperbolas are e_1 and e_2, show that $e_1^2+e_2^2 = e_1^2 e_2^2$.

22. Show that the equations $x = at + \frac{b}{t}$, $y = bt - \frac{a}{t}$ represent a rectangular hyperbola, t being variable and the axes of coordinates rectangular. [J.]

23. Prove that the asymptotes of a hyperbola meet the directrices on the auxiliary circle.

24. The tangent at the point (x', y') on the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2$$

cuts the auxiliary circle in points whose ordinates are y_1 and y_2. Prove that $\frac{1}{y_1} + \frac{1}{y_2} = \frac{2}{y'}$.

25. Prove that the product of the perpendiculars from any point of a hyperbola on its asymptotes is constant. Hence deduce the equation of a hyperbola whose asymptotes are

$$x\cos\alpha + y\sin\alpha = p \quad \text{and} \quad x\cos\beta + y\sin\beta = q.$$

26. The tangents at the vertices of the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2$$

cut the tangent at any other point on the curve at T and T'. Prove that $\angle TST'$ and $\angle TS'T'$ are right angles.

27. If Q and Q' are two points on a rectangular hyperbola, and P is a third point on it such that $\angle QPQ'$ is a right angle, prove that QQ' is parallel to the normal at P.

28. If A, A' be the extremities of the transverse axis of a rectangular hyperbola and P be any point on the curve, show that $\angle PAA' \sim \angle PA'A$ is equal to a right angle.

29. A, A' being the vertices at the extremities of the transverse axis of a hyperbola and P any other point on it, prove that the bisectors of the angle APA' are parallel to the asymptotes.

30. Prove that the angles which a chord of a rectangular hyper-

bola subtends at the ends of any diameter are equal or supplementary.

31. AA' is the transverse axis of a hyperbola and P any other point on it. PA, PA' meet a directrix in Q and Q' respectively. Prove that QQ' subtends a right angle at the focus.

32. If a tangent to a hyperbola meets the asymptotes in L and L', and S and S' are the foci, prove that the points L, L', S, S' are concyclic.

33. The ordinate of any point P on a hyperbola meets the conjugate hyperbola in Q. Prove that the normals at P and Q to the respective hyperbolas meet on the transverse axis.

34. CP and CD are conjugate semidiameters of a rectangular hyperbola whose centre is C and foci S and S'. Prove that $CD^2 = SP.S'P$.

35. Show that the normal at a point $\left(c\mu, \dfrac{c}{\mu}\right)$ on the rectangular hyperbola $xy = c^2$ cuts the curve again in the point $\left(-\dfrac{c}{\mu^3}, -c\mu^3\right)$.

36. The normal at P to the rectangular hyperbola $xy = c^2$ meets the curve again at Q. Prove that $PQ = \dfrac{CP^3}{c^2}$, C being the centre.

[O. and C.]

37. Prove that a directrix is the polar of the corresponding focus of a hyperbola.

38. Prove that the polars of any point with regard to a hyperbola and its conjugate are parallel.

39. If two hyperbolas are conjugate to each other, prove that the polar of a point on one with respect to the other is a tangent to the former.

40. Prove that the polar of any point on an asymptote of a hyperbola is parallel to it.

41. Find the pole of the line $2x-3y = 108$ with regard to the hyperbola $x^2-4y^2 = 36$.

42. The pole of the line $27x-100y = c$ with respect to the hyperbola $9x^2-225y^2 = 2025$ lies on the circle $x^2+y^2 = 225$. Find c.

43. Find the equation of the tangents from the point $(1, 1)$ to the rectangular hyperbola $xy = 4$, and the angles they make with an asymptote.

44. Find the locus of the pole of the line $x \cos\alpha + y \sin\alpha = p$ with regard to a hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

45. Find the locus of the pole of a focal chord of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ with regard to its conjugate hyperbola.

46. Find the locus of the poles of tangents to the auxiliary circle of a hyperbola with respect to the hyperbola.

47. Find the locus of the poles of normal chords of a rectangular hyperbola.

48. Prove that the line joining the poles of two parallel chords of a hyperbola is a diameter of it.

49. The tangent at P to a rectangular hyperbola cuts the conjugate hyperbola at Q and R. Prove that $PQ = PR$ and also that the pole of QR with respect to the conjugate hyperbola is at P', where PCP' is a diameter of the original hyperbola.

50. Show that the locus of a point, which moves so that its polars with respect to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ and its auxiliary circle are perpendicular, is the pair of asymptotes of the hyperbola.

51. Find the locus of the poles of tangents to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ with respect to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

52. The normal at any point P on the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ cuts the transverse axis at Q. NP, the ordinate of P, when produced, cuts an asymptote at P'. Prove that QP' is perpendicular to an asymptote.

53. Find the condition that the line $lx + my = 1$ may be a normal to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

54. Show that if CP and CD are conjugate diameters of a hyperbola, D being on the conjugate hyperbola, then PD is parallel to an asymptote.

55. PP' is a diameter of the rectangular hyperbola $xy = c^2$ and the tangent at P meets the asymptotes at Q and R. Prove that $P'Q$ and $P'R$ are tangents to the rectangular hyperbola $xy = -\frac{c^2}{8}$.

[O. and C.]

56. The normal to a rectangular hyperbola at P meets the asymptotes in Q and R and the transverse axis in G. Prove that $PG^2 = PQ.PR$.

57. Find the locus of the middle points of the portions of the tangents to a hyperbola intercepted between the axes.

58. From points on its auxiliary circle tangents are drawn to a rectangular hyperbola. Find the locus of the middle points of the chords of contact.

59. Prove that the locus of the centres of the circles circumscribing the triangle formed by any tangent to the hyperbola $b^2x^2-a^2y^2 = a^2b^2$ and its asymptotes is

$$4(a^2x^2-b^2y^2) = (a^2+b^2)^2.$$

60. Prove that the chords of a rectangular hyperbola $xy = c^2$ which subtend a right angle at a fixed point $\left(cp, \dfrac{c}{p}\right)$ on the curve are parallel, and that the locus of their middle points is the diameter $y+p^2x = 0$.

61. The polar with respect to the hyperbola $b^2x^2-a^2y^2 = a^2b^2$ of any point on the ellipse $b^2x^2+a^2y^2 = a^2b^2$ meets the asymptotes at P and Q. Show that the equation of the locus of the middle point of PQ is

$$(b^2x^2-a^2y^2)^2 = a^2b^2(b^2x^2+a^2y^2).$$

62. Find the equation of the axes of the hyperbola whose equation is $15x^2-17xy-4y^2 = 1$.

63. A rectangular hyperbola passes through the origin and its asymptotes are the lines $2y-x = 1$ and $y+2x = 3$; find its equation.

64. Find the equation of the rectangular hyperbola with the same foci as the ellipse $b^2x^2+a^2y^2 = a^2b^2$.

65. Find the equations of the asymptotes and of the conjugate hyperbolas of the hyperbolas whose equations are:

(1) $6x^2-7xy+2y^2+19x-11y-9 = 0.$
(2) $4x^2-24xy+11y^2+28x-94y-4 = 0.$

66. Find the tangent of the angle at which the rectangular hyperbolas $x^2-y^2 = 25$ and $xy = 4^2$ intersect.

67. Find the equations of the common tangents of:

(1) the hyperbolas $\frac{x^2}{25}-\frac{y^2}{9}=1$ and $\frac{y^2}{25}-\frac{x^2}{9}=1$;

(2) the parabola $y^2=4x$ and the rectangular hyperbola $y^2-x^2=4$.

68. The tangent to the rectangular hyperbola $xy=c^2$ at any point meets the asymptotes in P and P', and the normal at the same point cuts them in Q and Q'. If PQ' and $P'Q$ meet at R, prove that the locus of R is

$$4c^2(x^2-y^2)^2+xy(x^2+y^2)^2=0.$$

69. PSQ is a focal chord of a conic, the focus S lying between P and Q. The tangents at P and Q meet at T. Prove that the angle PTQ is acute, right, or obtuse according as the conic is an ellipse, parabola, or hyperbola. [O.]

70. If the sum of the ordinates of four points on a rectangular hyperbola is constant, show that the normals at these points will meet on a fixed straight line parallel to the transverse axis.

71. Three normals are drawn to a rectangular hyperbola from any point on it, other than the normal at the point itself. Prove that the centroid of the triangle formed by the three feet of these normals is the centre of the hyperbola.

72. Show that four points on a rectangular hyperbola are conormal if the sum of the products of their ordinates taken two at a time and of their abscissae taken in the same way is zero.

73. P and Q are any two points on a rectangular hyperbola. The circle on PQ as diameter cuts the curve again at R and S. Prove that RS is a diameter of the hyperbola.

74. If a circle intersects a rectangular hyperbola in the four points P, Q, R, S, show that the centre of mean position of these points bisects the line joining the centres of the circle and of the hyperbola.

75. Tangents are drawn to the rectangular hyperbola $xy=c^2$ from the point (x_1, y_1). Prove that the area they form with either asymptote is

$$2c(c^2-x_1y_1)^{\frac{1}{2}}.$$

76. The normals at A, B, C, D on a rectangular hyperbola meet at R. Prove that AB and CD are perpendicular, and that if these

chords cut the asymptotes in the points P, P' and Q, Q' respectively, the coordinates of R are $(OP+OP', OQ+OQ')$.

77. Two straight lines X_1Y_1, X_2Y_2, each of given length, cut two perpendicular axes in X_1, X_2 and Y_1, Y_2 so that the four points X_1, Y_1, X_2, Y_2 are concyclic. Prove that the locus of the centres of the circles for different positions of the lines is a rectangular hyperbola.

78. A circle having as a chord a diameter PCP' of a rectangular hyperbola cuts it again in QQ'. Show that QQ' will also be a diameter of the hyperbola.

79. A tangent to a rectangular hyperbola $xy = c^2$ meets the asymptotes in Q and R respectively. The lines through Q and R parallel to the asymptotes meet the hyperbola at H and K. Prove that the envelope of HK is the rectangular hyperbola $16xy = 25c^2$.

80. A chord PQ of a rectangular hyperbola when produced makes with the asymptotes a triangle of constant area. Prove that PQ envelops at its middle point another rectangular hyperbola.

81. Prove that the locus of the centre of a circle which touches externally two fixed circles is a hyperbola.

82. Find the polar equation of a rectangular hyperbola with respect to its centre as pole and an asymptote as initial line.

83. Taking the centre of a hyperbola as pole and the transverse axis as initial line, find the polar equation of the locus of the foot of the perpendicular drawn from the centre to any tangent.

84. Find the locus of the middle points of focal radii of a hyperbola.

85. Show that the equations $\frac{l}{r} = 1-e\cos\theta$, $\frac{l}{r} = -1-e\cos\theta$ represent the same conic.

86. The point P on a hyperbola whose focus is S is such that the tangent at P and the latus rectum through S meet on an asymptote. Prove that SP is parallel to the other asymptote.

87. A fixed hyperbola has a focus at S, and K is a fixed point on its transverse axis. XL is a fixed straight line perpendicular to this axis. A variable chord through K cuts the hyperbola at points P and Q. The bisector of $\angle PSQ$ cuts XL in L, and LM is drawn perpendicular to SP. Prove that SM is of constant length.

88. PQ is any chord of a rectangular hyperbola which subtends

a constant angle α at the focus S. SY is drawn perpendicular to PQ. Find the locus of Y.

89. Prove that perpendicular focal chords of a rectangular hyperbola are of equal length.

90. Show that the polar equation of the envelope of a family of circles which pass through the origin and have their centres on the rectangular hyperbola $r^2\cos 2\theta = a^2$ is $r^2 = 4a^2\cos 2\theta$. [B.]

Formulae of Chapter VII for reference

[Those identical with the ellipse, except that $-b^2$ is put for b^2, are omitted.]

1. Equation of chord: $\dfrac{x}{a}\cos\dfrac{\varphi-\varphi'}{2}-\dfrac{y}{b}\sin\dfrac{\varphi+\varphi'}{2}=\cos\dfrac{\varphi+\varphi'}{2}$.

2. Equations of tangent: $\dfrac{x}{a}-\dfrac{y}{b}\sin\varphi=\cos\varphi$.

$$x+y\mu^2=2c\mu.$$

$$\frac{x}{x'}+\frac{y}{y'}=2.$$

3. Equations of normal: $ax\sin\varphi+by=(a^2+b^2)\tan\varphi$.

$$c\mu^4-\mu^3x+\mu y-c=0.\dagger$$

$$xx'-yy'=x'^2-y'^2.\dagger$$

4. Equations of asymptotes: $y=\pm\dfrac{b}{a}x$.

5. Equations of conjugate hyperbola: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=-1$.

$$xy=-c^2.$$

6. Equation of polar of (h,k): $\dfrac{x}{h}+\dfrac{y}{k}=\dfrac{2c^2}{hk}$.

7. Equation of tangents from (h,k):

$$(xk+yh-2c^2)^2=4(xy-c^2)(hk-c^2).$$

8. Tangents at μ_1,μ_2 on $xy=c^2$ meet at the point:

$$\frac{2c\mu_1\mu_2}{\mu_1+\mu_2},\ \frac{2c}{\mu_1+\mu_2}.$$

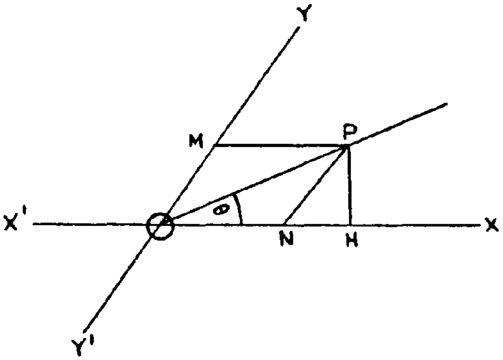
† For a rectangular hyperbola only.

VIII

OBLIQUE AXES

1. Equation of a straight line

In all the preceding chapters the axes of coordinates have been taken as rectangular. There are many cases, however, where work can be greatly simplified by taking as axes two lines which are not perpendicular. Such axes are known as *Oblique Axes*, and the acute angle between them is usually denoted by the Greek letter ω.



In the above diagram XOX', YOY' are oblique axes making an angle $XOY = \omega$ with each other. Through any point P draw PM, PN parallel to these axes and cutting them in N and M respectively.

Then ON and OM are the coordinates of P referred to these axes.

It is clear that many of the results arrived at hitherto can be proved by identical methods when the axes are oblique.

For example, the proof that any equation of the first degree represents a straight line would still be valid.

In the diagram draw PH perpendicular to OX.

Consider the equation $y = nx$.

It is of the first degree and therefore represents some straight line. Also, as (0, 0) satisfies it, it is a line through O.

If P be any point on it, $\dfrac{y}{x} = \dfrac{PN}{ON}$, which, from elementary geometrical considerations, must evidently be the same ratio, no matter where P is on the line. Calling this constant ratio n, we see that any equation such as $y = nx$ represents a straight line through the origin whose direction will only depend upon the value of n.

n is not the gradient of the line, but it can be expressed in terms of it thus, if $\angle POX = \theta$:

$$\tan\theta = \tan P\widehat{O}N = \frac{PH}{OH} = \frac{PN \sin P\widehat{N}H}{ON + NH}$$

$$= \frac{y \sin\omega}{x + y\cos\omega} = \frac{\dfrac{y}{x}\sin\omega}{1 + \dfrac{y}{x}\cos\omega} = \frac{n\sin\omega}{1 + n\cos\omega}.$$

Hence if the gradient of the line is m,

$$\mathbf{m} = \frac{\mathbf{n \sin\omega}}{\mathbf{1 + n\cos\omega}}.$$

It is clear, then, that if two lines have the same 'm' they will have the same 'n'.

$\therefore$ with oblique axes, $y = nx$, $y = n'x$ are parallel if $n = n'$.

2. Angle between two straight lines

Suppose the straight lines are respectively parallel to $y = nx$, $y = n'x$, and that the gradients of these lines are m and m'.

Then if ϕ be the angle between the two lines,

$$\tan\phi = \frac{m - m'}{1 + mm'} = \frac{\dfrac{n\sin\omega}{1 + n\cos\omega} - \dfrac{n'\sin\omega}{1 + n'\cos\omega}}{1 + \dfrac{nn'\sin^2\omega}{(1 + n\cos\omega)(1 + n'\cos\omega)}}$$

$$= \frac{(n - n')\sin\omega}{1 + (n + n')\cos\omega + nn'}.$$

This result shows that the lines will be parallel, i.e. $\phi = 0$ if $n = n'$.

They will be perpendicular if $\cot\phi = 0$, i.e. if

$$1+(n+n')\cos\omega+nn' = 0.$$

If ω were 90°, i.e. if the axes were rectangular, n and n' would then be the gradients of the lines and $nn' = -1$.

3. Some typical cases of linear equations

It may first be noted that, in transferring to parallel axes through another origin, (h, k) suppose, it is still true that the coordinates of a point which are (x, y) with regard to the new axes will be $(x+h, y+k)$ with regard to the old ones, i.e. we substitute $x+h$ for x and $y+k$ for y when transforming the original equation.

(1) Two straight lines $y = nx$, $y = n'x$ make equal angles with the x-axis; show that

$$2\cos(\pi-\omega) = \frac{1}{n}+\frac{1}{n'}.$$

By data $$\frac{n\sin\omega}{1+n\cos\omega} = -\frac{n'\sin\omega}{1+n'\cos\omega}.$$

Whence $$n+n' = -2nn'\cos\omega,$$

and $$\frac{1}{n}+\frac{1}{n'} = 2\cos(\pi-\omega).$$

[N.B. Two lines $y = nx$, $y = -nx$ in oblique coordinates do not make equal angles with the x-axis.]

(2) H and K are two fixed points and P is a point such that PH, PK cut off an intercept of fixed length on a given straight line. Find the locus of P.

Here it is obviously an advantage to take the fixed line as $y = 0$, and the line joining the two given points as $x = 0$.

Let P be the point (X, Y) referred to these oblique axes.

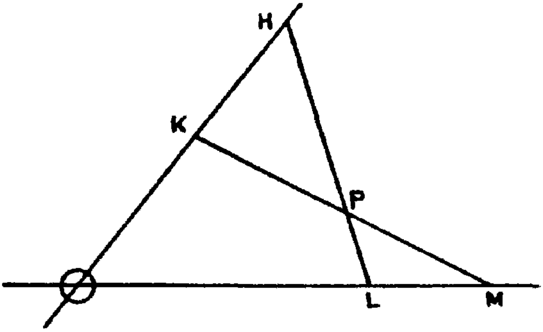
Any line through P is $y-Y = n(x-X)$.

If it passes through H, the point $(0, h)$, we have

$$h - Y = n(-X);$$

whence the equation of PH is

$$\frac{x-X}{X} = \frac{y-Y}{Y-h}.$$



Similarly, if K be the point $(0, k)$, the equation of PK is

$$\frac{x-X}{X} = \frac{y-Y}{Y-k}.$$

Where these lines cut $y = 0$ we have

$$OL = x_1 = X - \frac{XY}{Y-h},$$

$$OM = x_2 = X - \frac{XY}{Y-k}.$$

But as by data $LM = OM - OL = c$, a constant,

$$\therefore\ c = \frac{XY}{Y-h} - \frac{XY}{Y-k}$$

or $$c(Y-h)(Y-k) = XY(h-k)$$

is the equation of the locus required.

(3) **Find the locus of the middle point of any straight line which passes through a fixed point and has its extremities on two fixed straight lines.**

Take the two fixed straight lines as oblique axes of coordinates. Let (h, k) be the coordinates of the fixed point.

The equation of any line through (h, k) is

$$y-k = n(x-h).$$

Where this cuts $y = 0$, $x = h-\frac{k}{n}$.

Where it cuts $x = 0$, $y = k-nh$.

If (X, Y) be the middle point of the line, we have

$$2X = h-\frac{k}{n} \quad \text{and} \quad 2Y = k-nh.$$

$$\therefore \quad (2X-h)(2Y-k) = \left(-\frac{k}{n}\right)(-nh) = hk,$$

which is a hyperbola with its centre at the point $\left(\frac{h}{2}, \frac{k}{2}\right)$ and asymptotes parallel to the two given straight lines.

(4) **A line moves so as always to form a triangle of given area with two fixed lines. Find the equation of its envelope.**

Taking the fixed lines as axes of coordinates, let the equation of the moving line be $lx+my = 1$.

Where it cuts $x = 0$, $y = 0$ we have $y = \frac{1}{m}$, $x = \frac{1}{l}$.

And by data $\frac{1}{2}\frac{1}{l}.\frac{1}{m}\sin\omega = \text{constant}$.

$$\therefore \; lm = \text{constant} = K, \text{ say}.$$

So the equation of the line can be rewritten

$$lx+\frac{K}{l}y = 1,$$

i.e.
$$l^2x-l+Ky = 0,$$

of which the envelope is $4Kxy = 1$, this being the condition for equal roots of the quadratic in the variable. The line will therefore always touch a fixed hyperbola.

(5) In any triangle ABC, with the usual notation, if C is given and $\frac{1}{a}+\frac{1}{b} = K$, a constant, prove that the base AB must always pass through a fixed point.

Taking CA, CB as axes, let $lx+my = 1$ be the equation of the base AB.

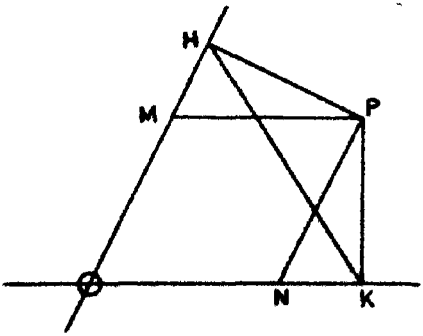
We are given that $\frac{1}{a}+\frac{1}{b} = K$. $\therefore\ l+m = K$.

$$\therefore\ l\left(\frac{1}{K}\right)+m\left(\frac{1}{K}\right) = 1;$$

which shows that the base AB always passes through the fixed point $\left(\frac{1}{K}, \frac{1}{K}\right)$.

(6) If the sum of the perpendiculars from a point P to two fixed lines be constant, find the locus of P.

Take the two fixed lines as axes of coordinates; let them be inclined at an angle ω to each other.



Draw PM, PN parallel to the axes, to cut them in M and N. Draw PH, PK perpendicular to the axes.

Let the coordinates of P be (X, Y).

From the diagram

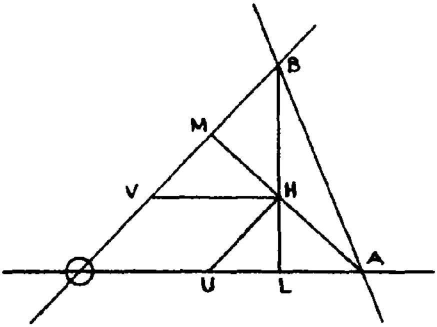
$$PH = PM \sin\omega = X \sin\omega,$$

$$PK = PN \sin\omega = Y \sin\omega.$$

$$\therefore\ (X+Y)\sin\omega = PH+PK = c, \text{ a constant.}$$

Hence the locus is a straight line which would always pass through the fixed point $\frac{c}{2}\operatorname{cosec}\omega, \frac{c}{2}\operatorname{cosec}\omega$.

(7) A straight line, of fixed length, slides between two given fixed straight lines inclined at an angle ω. Find the locus of the orthocentre of the triangle it forms with them.



Let AB be the line $\frac{x}{\alpha}+\frac{y}{\beta}=1$.

Draw AM, BL perpendicular to OB and OA respectively and let them intersect in H the point (X, Y).

Through H draw HU, HV parallel to the axes, so that $X = HV$ and $Y = HU$.

Then, since $HU : OM = AU : OA$,

$$\therefore\ \frac{Y}{\alpha\cos\omega} = \frac{\alpha - X}{\alpha}.$$

Similarly,
$$\frac{X}{\beta\cos\omega} = \frac{\beta - Y}{\beta}.$$

$$\alpha = X + Y\sec\omega \quad \text{and} \quad \beta = Y + X\sec\omega.$$

But if l be the fixed length of AB,

$$\begin{aligned} l^2 &= \alpha^2 + \beta^2 - 2\alpha\beta\cos\omega \\ &= (X + Y\sec\omega)^2 + (X\sec\omega + Y)^2 \\ &\qquad - 2(X + Y\sec\omega)(Y + X\sec\omega)\cos\omega, \end{aligned}$$

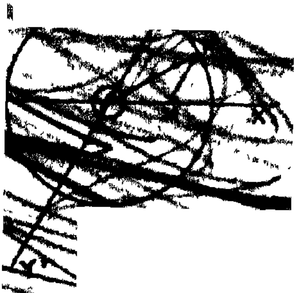
i.e.
$$(X^2 + Y^2)\tan^2\omega + 2XY(\sec\omega - \cos\omega) = l^2,$$

which is the required locus.

4. The circle

Let P be a point (x, y) on a circle referred to a pair of diameters as axes of coordinates which are inclined to each other at an angle ω.

From P draw a straight line PM parallel to the diameter YOY' to cut the diameter XOX' in M. Join OP.

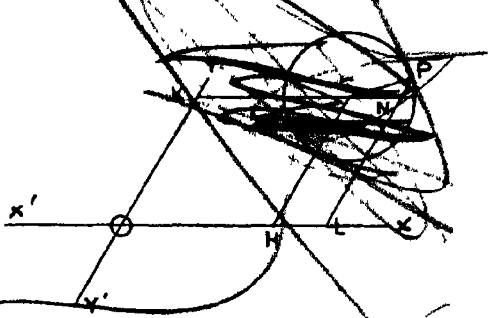


Then
$$\begin{aligned} a^2 = OP^2 &= OM^2 + PM^2 - 2OM \, . \, PM\cos\angle PMO \\ &= x^2 + y^2 - 2xy\cos(180^\circ - \omega) \\ &= x^2 + y^2 + 2xy\cos\omega. \end{aligned}$$

Let the centre C of a circle now be the point (h, k) referred to oblique axes XOX', YOY'. Also let $P (x, y)$ be any point on the circumference of this circle.

Draw CH, CK through C parallel to the y- and x-axes respectively.

Draw PNL parallel to the y-axis cutting the x-axis in L and KC produced in N.



Then $a^2 = CP^2 = CN^2+PN^2-2CN \,.\, PN \cos \angle PNC$

$$= (x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega.$$

Hence

$$x^2+2xy\cos\omega+y^2-2x(h+k\cos\omega)-2y(k+h\cos\omega)$$
$$+h^2+k^2+2hk\cos\omega-a^2 = 0.$$

We observe that in this equation the coefficients of x^2 and y^2 are equal, but there is always an xy term whose coefficient is $2\cos\omega$ times that of x^2 or y^2.

If the equation of the circle in oblique coordinates be written x^2+2xy [illegible]

the coordinates of the centre can [illegible] tions

$$-g = h+k\cos\omega \quad \text{and} \quad -f = \text{[illegible]},$$

whence $\quad h = \dfrac{f\cos\omega-g}{\sin^2\omega} \quad$ and $\quad k = \dfrac{g\cos\text{[illegible]}}{\sin\text{[illegible]}}$

The radius is the square root of

$$h^2+k^2+2hk\cos\omega-c$$

$$=\frac{1}{\sin^4\omega}[(f\cos\omega-g)^2+(g\cos\omega-f)^2$$

$$+2(f\cos\omega-g)(g\cos\omega-f)\cos\omega]-c$$

$$=\frac{f^2+g^2-2fg\cos\omega}{\sin^2\omega}-c.$$

The following problem gives an illustration of the advantageous use of oblique coordinates in the case of a circle.

A circle touches one fixed line and makes an intercept of constant length l on another fixed line inclined to the former at an angle ω. Find the locus of its centre.

Taking the fixed lines as oblique axes, let the equation of the circle be

$$x^2+2xy\cos\omega+y^2-2(h+k\cos\omega)x-2(k+h\cos\omega)y$$

$$+h^2+k^2+2hk\cos\omega-a^2=0.$$

Where this cuts $y=0$,

$$x^2-2(h+k\cos\omega)x+h^2+k^2+2hk\cos\omega-a^2=0,$$

and this equation must have equal roots if $y=0$ is a tangent to the circle.

$$\therefore\ (h+k\cos\omega)^2=h^2+k^2+2hk\cos\omega-a^2,$$

whence $$k^2\sin^2\omega=a^2.$$

Again, where the circle cuts $x=0$,

$$y^2-2(k+h\cos\omega)y+h^2+k^2+2hk\cos\omega-a^2=0.$$

If the roots of this equation are y_1, y_2, then

$$y_1+y_2=2(k+h\cos\omega),\quad y_1y_2=h^2+k^2+2hk\cos\omega-a^2.$$

$$\therefore\ l^2=(y_1-y_2)^2=4(a^2-h^2\sin^2\omega)$$

$$=4\sin^2\omega(k^2-h^2).$$

$\therefore$ the required locus is

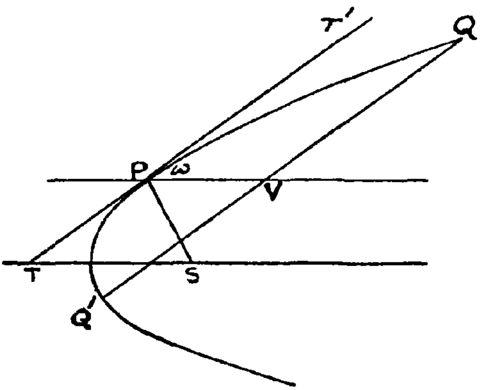
$$y^2-x^2=\tfrac{1}{4}l^2\operatorname{cosec}^2\omega.$$

5. The parabola

The most useful case for employing oblique axes in the case of a parabola is when it is desirable to take a diameter and tangent at a point on it as axes of coordinates.

Let QVQ' be any chord of a parabola parallel to the tangent at a point P on it and cutting the diameter through P in V. Join SP.

Then, as was proved in Chapter V, Art. 13, V is the middle point of QQ' and $QV^2 = 4SP.PV$.



Also $SP = a \operatorname{cosec}^2\omega$, where the angle between the tangent and diameter at P is ω.

$\therefore$ the equation of the parabola can be written $\mathbf{y^2 = 4\alpha x}$, where $\alpha = a \operatorname{cosec}^2\omega$.

As the coordinates of any point on $y^2 = 4\alpha x$ can be taken to be αt^2, $2\alpha t$, the equation of the chord joining two points t_1, t_2 is

$$x - \tfrac{1}{2}(t_1+t_2)y + \alpha t_1 t_2 = 0$$

and that of the tangent at the point t is

$$x - ty + \alpha t^2 = 0.$$

The three following examples illustrate the use of the equation $y^2 = 4\alpha x$.

(1) Two tangents to a parabola cut off on a given diameter a length which always bears a fixed ratio to the difference of the lengths of the perpendiculars from their points of contact to the tangent at the extremity of the fixed diameter.

Taking the diameter and the tangent at its extremity as axes of coordinates, the equation of the parabola is $y^2 = 4ax$.

Let t_1, t_2 be the points of contact of the two tangents.

Where $x-t_1 y+at_1^2 = 0$ cuts $y = 0$, $x = -at_1^2$.

Where $x-t_2 y+at_2^2 = 0$ cuts $y = 0$, $x = -at_2^2$.

$\therefore$ the intercept on the diameter is $a(t_1^2-t_2^2)$.

The perpendiculars from the points of contact are $at_1^2 \sin \omega$ and $at_2^2 \sin \omega$, and the difference of these is

$$a \sin \omega (t_1^2-t_2^2),$$

which establishes the theorem, the ratio being $\operatorname{cosec} \omega$.

(2) To find the locus of a point which moves so that the tangents from it to a parabola always make an intercept of constant length on a fixed tangent.

Taking the fixed tangent as the y-axis and the diameter through its point of contact as the x-axis, the equation of the parabola is $y^2 = 4ax$.

Suppose t_1, t_2 are the points of contact of two tangents which satisfy the required condition and (X, Y) the point where they meet.

Then $\qquad X = at_1 t_2 \quad$ and $\quad Y = a(t_1+t_2)$.

Where the tangent at t_1, whose equation is $x-t_1 y+at_1^2 = 0$, cuts $x = 0$, we have $y = at_1$.

So where the tangent at t_2 cuts $x = 0$, $y = at_2$.

We are given that $a(t_1-t_2) = l$, a constant. Hence

$$l^2 = a^2(t_1-t_2)^2 = a^2(t_1+t_2)^2-4a^2t_1 t_2$$
$$= Y^2-4aX.$$

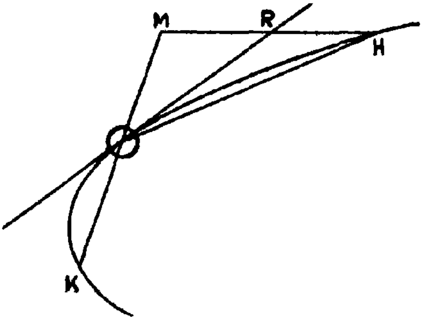
$\therefore$ the required locus is

$$Y^2 = 4a\left(X+\frac{l^2}{4a}\right),$$

which is the equation of another parabola similar to the original one and with an equal latus rectum.

(3) Two chords OH, OK of a parabola are drawn equally inclined to the tangent at O, a point on it. The diameter through H cuts KO produced in M. Show that

$$OM^2 = OH.OK.$$



Let the tangent at O cut HM in R.

Let the equation of the parabola referred to the diameter and tangent at O as axes be $y^2 = 4ax$.

Let H be the point $(at^2, 2at)$, K the point $(at'^2, 2at')$.

Then
$$\frac{OM}{OK} = \frac{2at}{2at'} = \frac{t}{t'}.$$

Also, as OR bisects $\angle MOH$,
$$\frac{OH}{OM} = \frac{HR}{RM}.$$

The equation of OK is $x = \frac{t'}{2}y$.

At M, where OK cuts the line $y = 2at$, we have
$$MR = x = att' \quad \text{and} \quad HR = at^2.$$
$$\therefore \frac{OH}{OM} = \frac{at^2}{att'} = \frac{t}{t'} = \frac{OM}{OK}.$$
$$\therefore OM^2 = OH.OK.$$

6. The ellipse

Here the most useful case to employ oblique axes is that in which a pair of conjugate diameters are taken as axes of coordinates.

The equation of an ellipse referred to its centre as origin and a pair of conjugate diameters as axes must be of the form

$$Ax^2+2Hxy+By^2 = 1.$$

The reason for this is that the curve being symmetrical about its centre, its equation must remain the same in form when $-x$ and $-y$ are substituted for x and y in it. Obviously this could not be the case were there terms such as $2Gx$ or $2Fy$ in it of the first degree in x and y.

Suppose the lengths of the conjugate diameters taken as axes of coordinates are $2a'$ and $2b'$.

When any line parallel to the y-axis such as $x = h$ cuts the ellipse it must be bisected by the x-axis, which is the locus of the middle points of all chords parallel to the y-axis.

This means that the algebraic sum of the two values of y is zero when $x = h$ is put in the equation.

$\therefore$ in the equation $By^2+2Hhy+Ah^2-1 = 0$ it follows that $2Hh = 0$. And as h is not zero, $\therefore$ $H = 0$. So the equation of the ellipse becomes $Ax^2+By^2 = 1$.

When $y = 0$, $x = a'$, and when $x = 0$, $y = b'$.

$$\therefore\ Aa'^2 = 1 = Bb'^2,$$

and the equation becomes

$$\mathbf{\frac{x^2}{a'^2}+\frac{y^2}{b'^2} = 1.}$$

Parametric coordinates $x = a'\cos\phi$, $y = b'\sin\phi$ can be taken for any point of the ellipse $\frac{x^2}{a'^2}+\frac{y^2}{b'^2} = 1$.

***The equations of the chord and of the tangent* will be the *same in form* and *arrived* at *by the* same method as when the axes were rectangular. It must be noted, however, that**

the parameter ϕ is no longer the eccentric angle of the point $(a'\cos\phi, b'\sin\phi)$ on the ellipse.

The following examples are illustrative:

(1) To show that if $y = nx$, $y = n'x$ are two conjugate diameters of the ellipse $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}=1$, then $nn' = -\dfrac{b'^2}{a'^2}$.

Suppose that the two points

$$(a'\cos\phi, b'\sin\phi),\ (a'\cos\phi', b'\sin\phi')$$

on the ellipse are the extremities of a chord parallel to $y = nx$.

The equation of the chord being

$$\frac{x}{a'}\cos\frac{\phi+\phi'}{2}+\frac{y}{b'}\sin\frac{\phi+\phi'}{2}=\cos\frac{\phi-\phi'}{2}.$$

$$\therefore\ n=-\frac{b'}{a'}\cot\frac{\phi+\phi'}{2}.$$

Let (X, Y) be the middle point of the chord, then

$$2X = a'(\cos\phi+\cos\phi') = 2a'\cos\frac{\phi+\phi'}{2}\cos\frac{\phi-\phi'}{2}$$

and $$2Y = b'(\sin\phi+\sin\phi') = 2b'\sin\frac{\phi+\phi'}{2}\cos\frac{\phi-\phi'}{2}.$$

$$\therefore\ \frac{Y}{X}=\frac{b'}{a'}\tan\frac{\phi+\phi'}{2}=-\frac{b'^2}{a'^2n}.$$

If this equation is the same as $y = n'x$, then

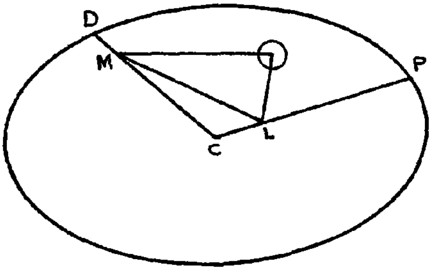
$$n' = -\frac{b'^2}{a'^2n}\quad\text{and}\quad nn' = -\frac{b'^2}{a'^2}.$$

(2) O is a fixed point in an ellipse and CP and CD are two conjugate diameters. Two lines are drawn through O parallel to any other pair of conjugate diameters to meet CP, CD in L and M respectively. Find the locus of the middle point of LM.

Let the equation of the ellipse referred to CP and CD as axes of coordinates be $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}=1$, and let O be the point (h, k).

Suppose that the two lines through O parallel to a pair of conjugate diameters are

$$y-k = n(x-h) \quad \text{and} \quad y-k = n'(x-h).$$



Then
$$nn' = -\frac{b'^2}{a'^2}.$$

At L, where the line $y = 0$ cuts the line $y-k = n(x-h)$,

$$CL = x = h-\frac{k}{n}.$$

And at M, where the line $x = 0$ cuts the line $y-k = n'(x-h)$,

$$CM = y = k-n'h.$$

If (X, Y) be the middle point of LM,

$$2X = h-\frac{k}{n} \quad \text{and} \quad 2Y = k-n'h.$$

$$\therefore\ n = \frac{k}{h-2X} \quad \text{and} \quad -n' = \frac{2Y-k}{h}.$$

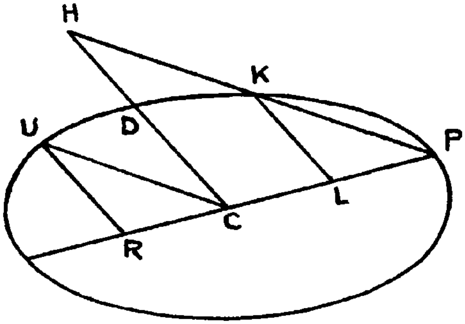
$$\therefore\ \frac{k}{h}\cdot\frac{2Y-k}{h-2X} = -nn' = \frac{b'^2}{a'^2}$$

or
$$ka'^2\left(Y-\frac{k}{2}\right) = hb'^2\left(\frac{h}{2}-X\right),$$

which is the equation of a straight line. It also clearly passes through the point $\left(\frac{h}{2}, \frac{k}{2}\right)$.

(3) If CP, CD are two conjugate semidiameters of an ellipse and any chord through P cuts CD (or CD produced) in H and the ellipse again in K, prove that $PH.PK = CU^2$, where CU is the semidiameter parallel to PK.

Draw UR, KL parallel to CD to meet CP (or CP produced) in R and L respectively.



The equation of the ellipse referred to CP and CD as axes of coordinates being $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}=1$, let K be the point $(a'\cos\phi, b'\sin\phi)$ on it and U the point $(a'\cos\theta, b'\sin\theta)$. Then

$$PL = PC-CL = a'-a'\cos\phi = 2a'\sin^2\frac{\phi}{2}.$$

The equation of PK joining the points 0, ϕ is

$$\frac{x}{a'}\cos\frac{\phi}{2}+\frac{y}{b'}\sin\frac{\phi}{2} = \cos\frac{\phi}{2}.$$

The equation of CU is

$$\frac{x}{a'\cos\theta} = \frac{y}{b'\sin\theta},$$

and as CU is parallel to PK, we have $\dfrac{b'}{a'}\tan\theta = -\dfrac{b'}{a'}\cot\dfrac{\phi}{2}$.

$$\therefore\ \tan\theta = -\cot\frac{\phi}{2} = \tan\left(\frac{\pi}{2}+\frac{\phi}{2}\right).$$

$$\therefore \; CR = a'\cos\theta = -a'\sin\frac{\phi}{2};$$

$$\therefore \; PC.PL = 2a'^2\sin^2\frac{\phi}{2} = 2CR^2.$$

From parallels

$$PL : PC : CR = PK : PH : CU.$$

$$\therefore \; PH.PK = 2CU^2.$$

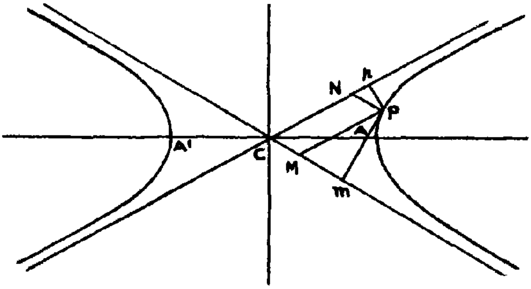
7. The hyperbola

This is a most important case in which the use of oblique axes is practically essential. In the previous chapter the rectangular hyperbola was considered with reference to its asymptotes as axes of coordinates. We shall now consider the case of any hyperbola so referred.

The equation of a hyperbola referred to its transverse and conjugate axes was found to be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Let P be any point on it and draw PM, PN parallel to the asymptotes, meeting them in M and N respectively.

Draw Pm, Pn perpendicular to the asymptotes, meeting them in m and n respectively.



Let P be the point whose coordinates are $a\sec\phi$, $b\tan\phi$. The equation of the asymptote CMm is $bx+ay=0$.

The perpendicular Pm on this line $= \dfrac{ab(\sec\phi + \tan\phi)}{\sqrt{(a^2+b^2)}}$.

Similarly, the perpendicular Pn on the asymptote

$$bx-ay = 0$$

is

$$\frac{ab(\sec\phi - \tan\phi)}{\sqrt{(a^2+b^2)}}.$$

$$\therefore\ Pm.Pn = \frac{a^2b^2(\sec^2\phi-\tan^2\phi)}{a^2+b^2} = \frac{a^2b^2}{a^2+b^2}.$$

Now if the angle between the asymptotes be 2α,

$$PM\sin 2\alpha = Pm \quad\text{and}\quad PN\sin 2\alpha = Pn.$$

$$\therefore\ PM.PN = Pm.Pn\operatorname{cosec}^2 2\alpha = \frac{a^2b^2}{a^2+b^2}.\operatorname{cosec}^2 2\alpha.$$

But $\tan\alpha = \dfrac{b}{a}$.

$$\therefore\ PM.PN = \frac{a^2b^2}{(a^2+b^2)}.\left(\frac{1+\tan^2\alpha}{2\tan\alpha}\right)^2$$

$$= \frac{a^2b^2}{(a^2+b^2)}.\left(\frac{1+\dfrac{b^2}{a^2}}{\dfrac{2b}{a}}\right)^2$$

$$= \frac{a^2+b^2}{4}.$$

Denoting $\dfrac{a^2+b^2}{4}$ by c^2, and now taking the asymptotes of the hyperbola as axes of coordinates, its equation derived from the relation just proved is

$$\mathbf{xy = c^2}.$$

The parametric coordinates of any point on it are $c\mu, \dfrac{c}{\mu}$. All equations, therefore, of the rectangular hyperbola found in the previous chapter hold for one which is not rectangular, except those, as, for example, the normal, where the perpendicularity relation $mm' = -1$ was brought into use.

We shall conclude this section with typical examples on the hyperbola whose equation is $xy = c^2$.

TYPICAL EXAMPLES

Ex. 1. Any chord PP' of a hyperbola cuts the asymptotes in p and p'. Prove that $Pp = P'p'$.

Let P, P' be the points μ, μ' on the hyperbola $xy = c^2$.

The equation of PP' is $x+y\mu\mu' = c(\mu+\mu')$.

At p, where it cuts $x = 0$, $y = c\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)$.

At p', where it cuts $y = 0$, $x = c(\mu+\mu')$.

$\therefore$ the middle point of pp' has coordinates

$$\frac{c}{2}(\mu+\mu'),\quad \frac{c}{2}\left(\frac{1}{\mu}+\frac{1}{\mu'}\right).$$

But these are also the coordinates of the middle point of PP'. $\therefore$ $pP = p'P'$.

Ex. 2. If from a point on the hyperbola $xy = c^2$ tangents are drawn to the conjugate hyperbola, the line joining their points of contact will touch the original hyperbola at a point which is at the other extremity of the diameter through the original point.

The polar of a point $\left(c\mu, \frac{c}{\mu}\right)$ with regard to $xy = -c^2$ is

$$\frac{x}{c\mu}+\frac{y\mu}{c} = -2,\quad \text{i.e. } x+y\mu^2 = -2c\mu,$$

which is the tangent to the original hyperbola at the point $-\mu$, the other extremity of the diameter through the point μ.

Ex. 3. The tangent at a point P of a hyperbola cuts the asymptotes in L and M. Find the locus of the centre of the circle circumscribing CLM, where C is the centre of the hyperbola.

Let the equation of the hyperbola be $xy = c^2$ and of the circle be $x^2+2xy\cos 2\alpha+y^2+2gx+2fy = 0$, where 2α is the angle between the asymptotes.

L is the point $(2c\lambda, 0)$ and M is the point $\left(0, \dfrac{2c}{\lambda}\right)$ if P is the point $\left(c\lambda, \dfrac{c}{\lambda}\right)$.

L and M are on the circle.

$$\therefore\ 4c^2\lambda^2+4gc\lambda = 0. \qquad \therefore\ g = -c\lambda.$$

And
$$\frac{4c^2}{\lambda^2}+\frac{4fc}{\lambda} = 0. \qquad \therefore\ f = -\frac{c}{\lambda}.$$

If (X, Y) be the coordinates of the centre of the circle,

$$X+Y\cos 2\alpha = -g = c\lambda$$

and
$$Y+X\cos 2\alpha = -f = \frac{c}{\lambda}.$$

$$\therefore\ (X+Y\cos 2\alpha)(Y+X\cos 2\alpha) = c^2$$

is the required locus.

Ex. 4. A quadrilateral has its four sides touching a hyperbola. Prove that the line joining the middle points of its diagonals AC, BD passes through the centre of the hyperbola.

Let A, B, C, D be the points where the tangents at the four points μ_1, μ_2, μ_3, μ_4 on the hyperbola $xy = c^2$ intersect.

Then A is the point $\dfrac{2c\mu_1\mu_4}{\mu_1+\mu_4}, \dfrac{2c}{\mu_1+\mu_4}$,

and C is the point $\dfrac{2c\mu_2\mu_3}{\mu_2+\mu_3}, \dfrac{2c}{\mu_2+\mu_3}$.

$\therefore$ the coordinates of the middle point of AC are

$$c\left[\frac{\mu_1\mu_4}{\mu_1+\mu_4}+\frac{\mu_2\mu_3}{\mu_2+\mu_3}\right],\quad c\left[\frac{1}{\mu_1+\mu_4}+\frac{1}{\mu_2+\mu_3}\right].$$

For brevity, let $\mu_1+\mu_2+\mu_3+\mu_4 = s_1$

and $\mu_1\mu_2\mu_3+\mu_1\mu_2\mu_4+\mu_1\mu_3\mu_4+\mu_2\mu_3\mu_4 = s_3$.

Then the coordinates of the middle point of AC are

$$\frac{cs_3}{(\mu_1+\mu_4)(\mu_2+\mu_3)},\ \frac{cs_1}{(\mu_1+\mu_4)(\mu_2+\mu_3)}.$$

$\therefore$ the equation of the line joining this point to the centre is

$$\frac{x}{s_3}=\frac{y}{s_1}.$$

Similarly, the coordinates of the middle point of BD are

$$\frac{cs_3}{(\mu_1+\mu_2)(\mu_3+\mu_4)},\ \frac{cs_1}{(\mu_1+\mu_2)(\mu_3+\mu_4)},$$

and the equation of the line joining this point to the centre is

$$\frac{x}{s_3}=\frac{y}{s_1}.$$

$\therefore$ the middle points of AC and BD and the centre of the hyperbola are collinear.

Ex. 5. If a circle cuts a hyperbola $xy = c^2$ in four points whose parameters are μ_1, μ_2, μ_3, μ_4, show that

$$\mu_1\mu_2\mu_3\mu_4 = 1.$$

The points of intersection of the circle

$$x^2+2xy\cos 2\alpha+y^2+2gx+2fy+k = 0$$

with the hyperbola $xy = c^2$ are obtained by putting $x = c\mu$, $y = \frac{c}{\mu}$ in the equation of the circle; whence we have

$$c^2\mu^2+\frac{c^2}{\mu^2}+2c^2\cos 2\alpha+2gc\mu+2f\frac{c}{\mu}+k = 0,$$

i.e. $$c^2\mu^4+2gc\mu^3+(2c^2\cos 2\alpha+k)\mu^2+2fc\mu+c^2 = 0,$$

from which we see that $\mu_1\mu_2\mu_3\mu_4 = 1$.

Ex. 6. C is the centre of a hyperbola $xy = c^2$ and P any point on it. On CP a point Q is taken such that $CP.CQ = k^2$, k being a constant. Find the locus of Q.

Let Q be the point (X, Y) and P the point $\left(c\lambda, \frac{c}{\lambda}\right)$.

Then $$\frac{X}{c\lambda}=\frac{CQ}{CP}=\frac{Y}{\frac{c}{\lambda}}. \qquad \therefore\ \frac{X}{Y}=\lambda^2.$$

Again, $$\frac{X}{c\lambda}=\frac{CQ.CP}{CP^2}=\frac{k^2}{c^2\lambda^2+\frac{c^2}{\lambda^2}}=\frac{k^2}{c^2\left(\frac{X}{Y}+\frac{Y}{X}\right)}.$$

$$\therefore\ \frac{k^4X^2Y^2}{c^4(X^2+Y^2)^2}=\frac{X^2}{c^2\lambda^2}=\frac{XY}{c^2},$$

i.e. $$k^4XY=c^2(X^2+Y^2)^2.$$

EXAMPLES 15

1. If the lines $y=nx+c$ and $y=5x$ are perpendicular, the angle between the axes being 45°, what is n?

2. Show that, if the axes of coordinates are inclined to each other at an angle ω, the two lines $y=3x-4$, $y=x+3$ will include an angle of $\tan^{-1}\frac{1}{2}\tan\frac{\omega}{2}$.

3. Find the equation of the line through the point $(3,4)$ perpendicular to the line $5x-3y=2$, the axes of coordinates being inclined at an angle of 60°.

4. The axes being inclined at an angle of 60°, find the equations of the lines through the origin which make 45° with the line $y=\sqrt{2}x$.

5. The axes being inclined at an angle of 30°, what is the gradient of the line $ax+by=1$?

6. Show that the perpendicular from a point (h,k) on the line $lx+my=1$ is of length

$$(lh+mk-1)\sin\omega/\sqrt{(l^2+m^2-2lm\cos\omega)}.$$

7. If $y=nx$ and $y=n'x$ make complementary angles with the x-axis, show that $1+(n+n')\cos\omega+nn'\cos 2\omega=0$.

8. PH, PK are perpendicular lines from a point $P\,(h,k)$ on the axes of coordinates which include an angle ω. Find the area of the triangle PHK in terms of h, k, and ω.

9. A line through a fixed point (h, k) cuts two fixed lines Ox, Oy at Q and R respectively. Find the locus of a point which divides QR in a constant ratio.

10. A line LM of fixed length moves with its extremities on two intersecting fixed lines. Show that the point on the line LM, which divides it in a fixed ratio λ, describes an ellipse.

11. A line through a fixed point (h, k) cuts two intersecting straight lines in points Q and R. The parallelogram $OQRT$ is completed. Find the locus of T.

12. A variable straight line cuts two fixed intersecting straight lines Ox, Oy in H and K so that the triangle OHK is of constant area. Find the envelope of HK.

13. If $l+m = c$, show that the line whose equation is $lx+my = 1$ always passes through a fixed point, and find its coordinates.

14. From a point P, perpendiculars PL, PM are drawn on two fixed intersecting straight lines. Find the locus of P if LM is always in a fixed direction.

15. A variable line passing through a fixed point cuts two fixed lines, which intersect at O, in points P and Q. Find the locus of the circumcentre of the triangle OPQ.

16. Two lines pass through fixed points H and K respectively and cut off a fixed intercept on another fixed straight line. Find the locus of their point of intersection.

17. Show that, if (x, y) are the coordinates of a point referred to axes Ox_1, Oy_1 inclined at an angle ω, and (x', y') the coordinates of the same point referred to axes Ox_2, Oy_2 inclined at an angle ω', then
$$x^2+y^2+2xy\cos\omega = x'^2+y'^2+2x'y'\cos\omega'.$$

18. Prove that if by any change of axes, the origin remaining the same, ax^2+by^2+2hxy becomes $a'x^2+b'y^2+2h'xy$, then
$$(a+b-2h\cos\omega)\sin^2\omega' = (a'+b'-2h'\cos\omega')\sin^2\omega$$
and
$$(ab-h^2)\sin^2\omega' = (a'b'-h'^2)\sin^2\omega,$$
by using the fact that if
$$ax^2+2hxy+by^2+\lambda(x^2+y^2+2xy\cos\omega)$$
is a perfect square, so must be
$$a'x^2+2h'xy+b'y^2+\lambda(x^2+y^2+2xy\cos\omega')$$
for the same value of λ.

19. Find the coordinates of the centre and the radius of the circle $x^2+2xy\cos 30^\circ+y^2-6x-2y = 0$.

20. The axes being inclined at an angle of 60°, find the equation of the circle whose centre is the point $(3, -1)$ and radius 9.

21. Show that the equation of a circle whose radius is a and which touches the axes of coordinates is

$$x^2+y^2+2xy\cos\omega-2a(x+y)\cot\frac{\omega}{2}+a^2\cot^2\frac{\omega}{2} = 0.$$

22. Write down the equation of the tangent to the circle $x^2+2xy\cos 60^\circ+y^2-6x-8y+19 = 0$ at the point $(2, 1)$.

23. Find the equation of the circle which intersects two fixed lines Ox, Oy at points H, K, where $OH = \alpha$ and $OK = \beta$, and which also passes through the origin.

24. Find the locus of the centre of a circle whose equation is $x^2+y^2+2xy\cos 60^\circ-2gx+10y+9 = 0$, where the axes are inclined at an angle of 60°, and g varies.

25. A variable circle touches two straight lines which intersect. Show that the polars of a fixed point with regard to these circles will envelop a conic.

26. A point moves so that the sum of the squares of its distances from the four vertices of a parallelogram is constant. Show that its locus is a circle.

27. The equation of a parabola referred to the diameter and tangent at a point O on it as axes of coordinates is $y^2 = 4\alpha x$. Find the equation of the chord joining two points $(\alpha t_1^2, 2\alpha t_1)$ $(\alpha t_2^2, 2\alpha t_2)$ on it, and that of the tangent at the point $(\alpha t^2, 2\alpha t)$.

28. The diameter through any point O on a parabola cuts any two tangents in L and M. Prove that, if the tangents meet at T, then $OL.OM = TR^2$, where R is the point in which a parallel through T to the diameter cuts the tangent at O.

29. Any chord PQ of a parabola cuts the diameter through a fixed point O on it in L, and the tangent at P cuts the diameter in T. Prove that $LP : LQ = TO : OL$.

30. If T be any point on the tangent at a fixed point O of a parabola, and any line through T cuts the parabola in H and K and the diameter through O in R, prove that $TH.TK = TR^2$.

31. Find the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

32. The diameter through a fixed point O on a parabola cuts any chord PQ in R. The tangents at P and Q meet the diameter in H, K. Show that $OR^2 = OH.OK$.

33. Two parabolas with parallel axes touch each other at a point O through which any line is drawn cutting the curves in P and Q respectively. Prove that $OP : OQ = SO : S'O$, where S and S' are the foci of the parabolas.

34. Two parabolas with parallel axes touch at a point O. Lines are drawn parallel to their common diameter through O. Prove that the locus of the middle points of the parts of them intercepted between the curves is a parabola.

35. Show that the length of any focal chord of a parabola is four times the focal distance of the extremity of the diameter bisecting it.

36. Prove that, if a parallelogram be inscribed in an ellipse, its diagonals must pass through the centre.

37. PCP' is a diameter of an ellipse and Q any point on the curve. The tangent at Q cuts the tangent at P in L, and $P'Q$ produced cuts it in M. Prove that $PL = LM$.

38. If the tangent to an ellipse at any point Q meets the diameter PCP' in T, and a chord through Q parallel to the conjugate diameter CD cuts PCP' in V, then $CV.CT = CP^2$. Prove also that $TC.TV = TP.TP'$.

39. The tangent at any point P on an ellipse meets two conjugate diameters in H and K. Show that

$$HP.PK = CD^2,$$

where CD is the conjugate semidiameter to CP.

40. CP, CD are conjugate semidiameters of an ellipse, and through P and D any two parallel chords are drawn cutting the curve in Q and R. Show that CQ and CR are conjugate semidiameters of the ellipse.

41. Prove that the straight line which joins the extremities of conjugate diameters of a hyperbola is parallel to one of its asymptotes and bisected by the other.

42. P and Q are fixed points on a hyperbola and R is a variable point on it. Show that RP and RQ intercept a constant distance on either asymptote.

43. P and Q are two fixed points on a hyperbola and R any variable point. PR, QR cut an asymptote in the points L and M respectively; show that LM is of constant length.

44. AL, AM are drawn through the vertex of a hyperbola parallel to the asymptotes, and any diameter cuts the curve in P and these lines in L and M. Prove that $CP^2 = CL.CM$.

45. Show that if $y = mx+k$ touches the hyperbola $xy = c^2$, then $k^2+4c^2m = 0$.

46. Find the equation of the director circle of the hyperbola $xy = c^2$.

47. Find the locus of the middle points of chords of the hyperbola $xy = c^2$ (1) which pass through a fixed point, (2) which are of a given length.

48. The tangent at a point P on a hyperbola cuts the asymptotes in Q and R. Show that the locus of the centre of the circle circumscribing the triangle OQR is

$$(x^2+y^2)\cos\omega+xy(1+\cos^2\omega) = c^2,$$

where ω is the angle between the asymptotes and O is the origin.

49. Find the locus of the middle point of the chord which any tangent to the parabola $x^2 = 4ay$ makes in the hyperbola $xy = c^2$.

50. Any four tangents to a hyperbola meet in points A, B, C, D. Prove that the middle points of AC and BD lie on a line which passes through its centre.

Formulae of Chapter VIII for reference

1. Gradient of $y = nx$ is: $\dfrac{\mathbf{n}\sin\omega}{1+\mathbf{n}\cos\omega}$.

2. If $y = nx$, $y = n'x$ are perpendicular:
$$1+(\mathbf{n}+\mathbf{n}')\cos\omega+\mathbf{n}\mathbf{n}' = 0.$$

3. Coordinates of the centre of the circle $x^2+2xy\cos\omega+y^2+2gx+2fy+c = 0$: $\dfrac{\mathbf{f}\cos\omega-\mathbf{g}}{\sin^2\omega}$, $\dfrac{\mathbf{g}\cos\omega-\mathbf{f}}{\sin^2\omega}$.

4. In the parabola $y^2 = 4\alpha x$: $\boldsymbol{\alpha = a\,\mathrm{cosec}^2\omega}$.

5. In the ellipse whose equation is

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

$y = nx$, $y = n'x$ are *conjugate* if: $\boldsymbol{nn' = -\frac{b'^2}{a'^2}}$.

6. If μ_1, μ_2, μ_3, μ_4 are four concyclic points on the hyperbola $xy = c^2$: $\boldsymbol{\mu_1\mu_2\mu_3\mu_4 = 1}$.

IX

ONE HUNDRED EXAMPLES FROM HIGHER CERTIFICATE EXAMINATION PAPERS SET BY VARIOUS EXAMINING BOARDS IN RECENT YEARS

1. *Prove that the equation of the straight line joining the points* (x_1, y_1), (x_2, y_2) is

$$(x-x_1)/(x_1-x_2) = (y-y_1)/(y_1-y_2).$$

$ABCD$ is a parallelogram and the coordinates of A, B, C, referred to rectangular axes, are respectively $(-8, -11)$, $(1, 2)$, $(4, 3)$. Find (i) the equations of AD and CD, (ii) the coordinates of D, and (iii) the area of the parallelogram $ABCD$. [L.]

2. The coordinates of the vertices of a triangle are $(2, 4)$, $(-1, 8)$, $(-1, -2)$. Find (1) the equations of the sides of the triangle, (2) its angles to the nearest minute, (3) its area. [J.]

3. Interpret the constants in the equation of a straight line in the form $y = mx+c$.

Two lines are drawn through the point $(0, c)$ making angle α_1, α_2 with the axis OX and meeting this axis at P_1 and P_2 respectively. If

$$\cot\alpha_1 + \cot\alpha_2 = k,$$

prove that the perpendiculars to the respective lines at P_1 and P_2 meet on a line parallel to the axis OY. [W.]

4. The coordinates of a moving point are given by

$$x = 2+\cos 2t,\ y = 1+\sin^2 t,$$

where t is variable. Prove that the locus of the point is part of a straight line.

Two adjacent vertices of a square slide on OX and OY respectively. In one position the square lies in the quadrant XOY. Find the equation of the locus of the centre of the square.

[O. and C.]

5. Find the angle between the lines

$$l_1x+m_1y+1 = 0,\quad l_2x+m_2y+1 = 0.$$

Find the equations of the two lines through the point $(3, -1)$ which make 45° with the line $2x = y+2$. [W.]

6. Two points P_1, P_2 have coordinates (x_1, y_1) and (x_2, y_2), and $ax+by+c = 0$ is a line which does not pass through either of them. Find the geometrical significance of the number

$$\frac{ax_1+by_1+c}{ax_2+by_2+c},$$

paying attention to the meaning of the sign.

Find the equation of the bisector of that angle between the lines $3x+4y-205 = 0$, $12x+5y+135 = 0$ within which the point $(-100, 250)$ lies. [C.]

7. Prove that the points $(a, bc-a^2)$, $(b, ca-b^2)$, $(c, ab-c^2)$ are collinear, and find the intercept on the axis of y by the straight line on which they lie. [O.]

8. Show that the equation of any straight line can be put in the form

$$x\cos\alpha + y\sin\alpha - p = 0,$$

and state the geometrical significance of α and p.

There are two segments, each of length 10 units, on the line

$$3x-4y+7 = 0$$

such that each segment subtends a right angle at the origin. Find the coordinates of their extremities. [B.]

9. Interpret the equations

$$\frac{x-a}{\cos\theta} = \frac{y-b}{\sin\theta} = r.$$

Find the distance from the point $(-1, 3)$ to the line

$$2x+3y = 1,$$

measured parallel to the line $7x-24y+3 = 0$. [C.]

10. Find the coordinates of the point which divides the straight line joining the points (x_1, y_1), (x_2, y_2) in the ratio $m_1 : m_2$.

Show that the straight line joining the points $(-3, 1)$, $(4, 2)$ is divided harmonically by the lines

$$x^2-8xy+2y^2 = 0. \qquad \text{[O.]}$$

11. Find the condition that the equation $ax^2+2hxy+by^2 = 0$ may represent two straight lines at right angles.

Show that the chord of the circle $x^2+y^2-2x-5y-24 = 0$ whose equation is $4x+3y = 24$ subtends a right angle at the origin. [D.]

12. Two straight lines are given by the equation

$$4y^2-8yx+3x^2 = 0.$$

Find the equation of a straight line whose gradient is $-\frac{3}{2}$ and which cuts the lines at A, B so that the area of the triangle OAB is 12 square units, O being the origin and A, B in the first quadrant. [B.]

13. Determine the equation of the straight line whose intercepts on the axes of x and y are a and b respectively.

If the line moves in such a way that

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2},$$

where c is a constant, prove that the foot of the perpendicular from the origin on the straight line describes the circle $x^2+y^2 = c^2$. [W.]

14. A right-angled isosceles triangle has one angular point at the origin and its hypotenuse along the line $2x-6y+15 = 0$. Find the equations of the other two sides.

Prove that the locus of a point which moves so that the sum of the squares of its distances from the vertices of this triangle is constant is a circle whose centre is the centroid of the triangle. [D.]

15. Show that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

represents two parallel straight lines provided that

$$\frac{a}{h}=\frac{h}{b}=\frac{g}{f},$$

and that, when these conditions are satisfied, the square of the distance between the lines is

$$\frac{4(g^2-ac)}{a(a+b)}.$$

Find also an expression for the length intercepted by these parallel lines on the line $y = mx$. [J.]

16. Find the value of k in order that the equation

$$12x^2-10xy+2y^2+11x-5y+k = 0$$

may represent a pair of straight lines.

If parallels are drawn to these two lines through the origin, find the area of the parallelogram so formed. [D.]

17. Prove that the equation

$$ax^2+2hxy+by^2 = 0$$

represents a pair of straight lines through the origin and that the sine of the angle between them is equal to

$$\frac{2\sqrt{(h^2-ab)}}{\sqrt{\{(a-b)^2+4h^2\}}}.$$

Prove that the length intercepted by these lines on the line $lx+my+n = 0$ is $\dfrac{2n\sqrt{\{(l^2+m^2)(h^2-ab)\}}}{am^2-2hlm+bl^2}$. [O. and C.]

18. Prove that

$$ax^2+2hxy+by^2-a(x-\xi)^2-2h(x-\xi)(y-\eta)-b(y-\eta)^2 = 0$$

is the equation of a diagonal of the parallelogram whose sides are $ax^2+2hxy+by^2 = 0$ and the parallels to these lines through (ξ, η).

Hence, or otherwise, prove that

(i) $m(ax+hy)-l(hx+by) = 0$ is the equation of the median, through the origin, of the triangle whose sides are

$$ax^2+2hxy+by^2 = 0 \quad \text{and} \quad lx+my-1 = 0;$$

(ii) $y(ax+hy)-x(hx+by) = 0$ is the equation of the bisectors of the angles between the lines

$$ax^2+2hxy+by^2 = 0.$$ [O.]

19. Show that the perpendicular distance of the point (α, β) from the line $ax+by+c = 0$ is $\pm\dfrac{a\alpha+b\beta+c}{\sqrt{(a^2+b^2)}}$, explaining clearly how the sign is determined.

The sides of a triangle ABC are given by the equations $AB : 4x+3y = 12$, $BC : y = \frac{5}{12}x+4$, $CA : 4y = 3x$.

Calculate the distance between the centres of the two escribed circles of the triangle which touch internally the sides AB and BC respectively. [L.]

20. Prove that the equation of the circle described on AB as diameter, where A is (x_1, y_1) and B is (x_2, y_2), is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0.$$

The straight line, $lx+my = 1$, meets the lines

$$ax^2+2hxy+by^2 = 0$$

in the points P and Q. Show that the equation of the circle on PQ as diameter is

$$(am^2-2hlm+bl^2)(x^2+y^2)-2(bl-hm)x-2(am-hl)y+a+b = 0.$$

[B.]

21. Prove that the locus whose equation referred to rectangular axes is $x^2+y^2+2gx+2fy+c = 0$ is a circle, and find the coordinates of its centre.

Prove that if the points (h_1, k_1), (h_2, k_2), (h_3, k_3) lie on the circle $x^2+y^2+2gx+2fy = 0$, then the four straight lines

$$\begin{aligned} h_1x+k_1y-m(h_1^2+k_1^2) &= 0,\\ h_2x+k_2y-m(h_2^2+k_2^2) &= 0,\\ h_3x+k_3y-m(h_3^2+k_3^2) &= 0,\\ fx-gy &= 0, \end{aligned}$$

meet in a point. [O.]

22. Find the condition that the circle

$$x^2+y^2+2g_1x+2f_1y+c_1 = 0$$

shall bisect the circumference of the circle

$$x^2+y^2+2g_2x+2f_2y+c_2 = 0.$$

Show that all circles passing through the two points $(-3, 0)$ and $(7, 0)$ bisect the circumferences of each of the circles

$$x^2+y^2+4x-5 = 0 \quad \text{and} \quad x^2+y^2-8x-5 = 0.$$

[L.]

23. Prove that the circle with centre at (a, b) and radius r lies wholly inside the circle with centre at (c, d) and radius s, provided that $s > r$ and also

$$(s-r)^2 > (c-a)^2+(d-b)^2.$$

Verify that these conditions are satisfied in the case of the circles

$$\begin{aligned} 4(x^2+y^2)+20x+4y+17 &= 0,\\ 4(x^2+y^2)+16x-4y-11 &= 0; \end{aligned}$$

and determine by calculation whether the centre of the larger circle lies inside or outside the smaller circle. [J.]

24. Prove that the equation of the tangent to the circle $x^2+y^2 = a^2$ at the point $(a\cos\theta, a\sin\theta)$ is

$$x\cos\theta + y\sin\theta = a.$$

If this tangent cuts the axes in the points M and N, prove that $MN^2 = a^2(\sec^2\theta+\operatorname{cosec}^2\theta)$, and that the least value of MN is $2a$. [B.]

25. Show that the expression $x_1^2+y_1^2+2gx_1+2fy_1+c$ represents the square of the tangent from the point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c = 0$.

Prove that the squares of the tangents drawn from any point on the circle $x^2+y^2+2x-16y = 0$ to the two circles

$$x^2+y^2-10y+2\lambda(x-3y) = 0,$$
$$x^2+y^2-10y+2\lambda'(x-3y) = 0$$

are in the constant ratio $\lambda-1 : \lambda'-1$. [O. and C.]

26. Write down the equations of the two circles which touch the lines $x = a$, $x = 3a$, $y+a = 0$.

The line which joins the origin to the point of contact of these two circles meets them again in P and Q. If the tangents at P and Q cut the axis of x in D and E, prove that $DE = a$. [D.]

27. A circle cuts the axis of x at points distant $2l$ apart. Obtain its equation in the form

$$(x-a)^2+y^2-2by-l^2 = 0.$$

Find the equation of the locus of the centre of a circle which passes through the fixed point $(0, c)$ and cuts the axis of x at points a fixed distance $2l$ apart. [J.]

28. Find the equation of the two straight lines joining the origin to the points of intersection of the circle

$$x^2+y^2+2gx+2fy+c = 0$$

and the straight line $lx+my = 1$.

Hence, or otherwise, find the coordinates of the circumcentre of the triangle whose sides are

$$ax^2+2hxy+by^2 = 0, \quad lx+my = 1. \qquad [C.]$$

29. The coordinates, referred to rectangular axes, of the points A, B, C are $(2h, 0)$, $(0, 2p)$, $(0, -2q)$ respectively. Find the equation to the circle through the origin and the middle points of AB and AC. Show that this equation is satisfied by the coordinates of the middle point of BC and by those of the middle point of the line joining A to the orthocentre of the triangle ABC.

State and prove the 'nine-point circle property' of any triangle. [O.]

30. Prove that the point $(a+r\cos\theta, b+r\sin\theta)$ lies on the circle $(x-a)^2+(y-b)^2 = r^2$, and find the equation of the tangent at the

point. Find the coordinates of the centres of similitude of the two circles $(x+a)^2+(y-b)^2 = r_1^2$ and $(x-a)^2+(y-b)^2 = r_2^2$. Show that the points of contact of the common tangents to the two circles lie on the circles

$$x^2+(y-b)^2 = a^2 \pm r_1 r_2. \qquad \text{[L.]}$$

31. What curves are represented by the equation

$$(x^2+y^2+2gx+2fy+c)+\lambda(lx+my+n) = 0,$$

in which λ is a variable parameter?

Find the equation of the circle which passes through the point $(-2, 4)$ and through the points in which the circle

$$x^2+y^2-2x-6y+6 = 0$$

is cut by the line $3x+2y-5 = 0$.

Find its centre and radius, and show that the two circles intersect at right angles.

Draw a careful figure. [W.]

32. Prove that the straight line $y = mx+c$ will touch the circle $(x-h)^2+y^2 = a^2$ if

$$m^2(h^2-a^2)+2chm+(c^2-a^2) = 0,$$

and deduce that the equation of the pair of tangents to the circle from the point $(0, c)$ is

$$(h^2-a^2)(y-c)^2+2ch(y-c)x+(c^2-a^2)x^2 = 0.$$

P being any point on the straight line $x = 0$, prove that the diagonal, other than $x = 0$, of the parallelogram formed by the tangents from P to the fixed circle $(x-h)^2+y^2 = a^2$, and the parallels to these tangents through the origin, passes through a fixed point. [O.]

33. Verify that the equation of the circumcircle of the triangle formed by the three lines $x+\theta y+\theta^2\kappa = 0$, where the parameter θ has the values a, b, c, which are the roots of the equation in t, $t^3-pt^2+qt-r = 0$ is

$$(x^2+y^2)-(1+q)\kappa x-(r-p)\kappa y+q\kappa^2 = 0.$$

Show that for all values of a, b, c this circle passes through a fixed point. [C.]

34. Obtain the equation of the polar of any point (x', y') with respect to the circle $x^2+y^2 = r^2$.

Find the locus of the pole of the fixed line $x = a$ with respect to the system of circles given by the equation

$$x^2+y^2+2fy+c = 0,$$

where c is fixed and f variable. [D.]

35. Show that the image of the point (x_0, y_0) in the line $lx+my+n = 0$ is given by

$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = -2\left(\frac{lx_0+my_0+n}{l^2+m^2}\right).$$

Prove that the image of the circle $x^2+y^2 = a^2$ in the same line is

$$(l^2+m^2)(x^2+y^2-a^2)+4n(lx+my+n) = 0.$$

[The image of a point P in a line is the point Q, where PQ is perpendicularly bisected by the line.] [W.]

36. Show that the circles

$$a_1x^2+a_1y^2+2g_1x+2f_1y+c_1 = 0,$$
$$a_2x^2+a_2y^2+2g_2x+2f_2y+c_2 = 0$$

are orthogonal if $2g_1g_2+2f_1f_2 = a_2c_1+a_1c_2$.

Find the locus of the centre of a circle which passes through the origin and cuts the circle $3x^2+3y^2+6x+9y+1 = 0$ orthogonally. [D.]

37. Prove that the circles

$$x^2+y^2+ax+by = 0, \qquad x^2+y^2 = c^2$$

touch each other if $a^2+b^2 = c^2$, and find the coordinates of the point of contact.

Two circles pass through the origin and the point $(1, 0)$ and touch the circle $x^2+y^2 = 4$. Find the coordinates of the points of contact. [O. and C.]

38. Show that, by suitable choice of the axes of coordinates, the equations of the circles of a coaxal system may be written in the form

$$x^2+y^2+k = 2\lambda x,$$

where k is constant, and λ varies from circle to circle of the system.

Show also that, if no two of the circles intersect in real points, there are always two real circles of the system which touch a given straight line

$$x\cos\alpha + y\sin\alpha = p,$$

and find the distance between the centres of these circles in terms of k, p, and α. [J.]

39. Find (i) the equation of the axis, (ii) the coordinates of the focus, (iii) the length of the latus rectum, of the parabola given by the equation $16x^2-24xy+9y^2-98x-64y-39 = 0$.

Give a rough sketch of this parabola. [C.]

40. If P and Q are the points in which the line $y = mx+c$ cuts the parabola $y^2 = 4ax$, find PQ in terms of a, m, and c.

Also, if M is the mid-point of PQ, and RM is drawn parallel to the axis of the parabola to cut the curve in R and to cut in M' a line drawn through the focus parallel to PQ, prove that

$$PQ^2 = 16RM.RM'. \qquad \text{[L.]}$$

41. The normal to a parabola at P cuts the curve again in Q. The tangents at P and Q meet at T; V is the middle point of PQ and S is the focus. Prove that the lines TV and PS intersect on the parabola. [J.]

42. Find in its simplest form the equation of the line joining the points $(am^2, 2am)$, $(at^2, 2at)$.

The chord PP' of a parabola is parallel to the directrix, and the lines joining a point Q of the curve to P and P' cut the axis of the parabola in R and R'. Show that the mid-point of RR' is the vertex. [C.]

43. Prove that the tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix.

A chord of a parabola is drawn through the point on the axis whose distance from the vertex is equal to the length of the latus rectum. Prove that the circle on this chord as diameter goes through the vertex of the parabola. [D.]

44. Through the point $P\,(\frac{3}{2}, 2)$ a line of gradient $\tan\theta$ is drawn to cut the parabola $y^2 = 2x$ at Q and R. Show that the lengths of PQ and PR are the roots of the quadratic

$$r^2\sin^2\theta+2r(2\sin\theta - \cos\theta)+1 = 0.$$

Hence, or otherwise, find the gradients of the tangents from P to the curve. [J.]

45. If the normal at the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$ meets the parabola again at the point $(at_1^2, 2at_1)$, find t_1 in terms of t.

PQ is a focal chord of a parabola and the normals at P and Q meet the parabola again at P' and Q' respectively. Prove that

$P'Q'$ is parallel to PQ and that the length of $P'Q'$ is three times that of PQ. [O.]

46. Prove that all chords of the parabola $y^2 = 4ax$ which pass through a certain point on the axis of the parabola subtend a right angle at its vertex.

A chord PP' subtends a right angle at the vertex A; the parallelogram $P'APR$ is completed; find the locus of R. [B.]

47. Find the condition that the straight line $y = mx+c$ shall be normal to the parabola $y^2 = 4ax$.

The tangents at points P, Q on the parabola meet in T, and the corresponding normals meet in R. If the locus of T is a straight line parallel to the axis of the parabola, prove that the locus of R is a straight line which is normal to the parabola. [L.]

48. Prove that the locus of the middle point of the chord intercepted by the parabola $y^2 = 4ax$ on any tangent to the parabola $y^2+4ax = 0$ is a parabola. [O.]

49. Taking the parametric representation of a point on the parabola $y^2 = 4ax$ in the form $(at^2, 2at)$, find the equations satisfied by the values of t belonging

(i) to the feet of the normals to the parabola from the point (h, k),

(ii) to the intersections of the parabola with the circle

$$x^2+y^2+2gx+2fy+c = 0.$$

Hence, or otherwise, prove that the circle through the feet of the normals passes through the vertex of the parabola, and that its equation is
$$x^2+y^2-(2a+h)x-\tfrac{1}{2}ky = 0. \qquad [\text{C.}]$$

50. Obtain the area of a triangle inscribed in the parabola $y^2 = 4ax$, in terms of the ordinates of its vertices.

Prove that the latus rectum of a parabola circumscribed about a triangle is equal to $2R\sin\theta_1\sin\theta_2\sin\theta_3$, where R is the radius of the circle circumscribed about the triangle and θ_1, θ_2, θ_3 are the angles which the sides of the triangle make with the axis of the parabola. [O. and C.]

51. Show that the circumcircle of the triangle formed by three tangents to a parabola passes through the focus.

The perpendicular bisector of a focal chord PSQ of a parabola meets the axis of the parabola in X; prove that $SX = \tfrac{1}{2}PQ$. [D.]

52. A circle cuts a parabola at four points, and from these points perpendiculars are drawn to the axis of the parabola. Prove that the sum of these perpendiculars is zero.

If the circle cuts the axis of the parabola at the focus S and at another point P, prove that the sum of the distances from S of the four points of intersection is $2SP$. [B.]

53. Prove that in a parabola the subtangent is twice the abscissa and that the subnormal is constant.

P, Q are two points on a parabola whose focus is S, and whose latus rectum is $4a$. Find the locus of the point of intersection of the tangents to the parabola at the points P and Q if $SP+SQ = 4a$. Show the relation of the locus to the parabola by means of a figure. [W.]

54. Prove that three real normals cannot be drawn from a point $(h, 0)$ on the axis of the parabola $y^2 = 4ax$ to the curve unless $h > 2a$.

Find the area of the triangle whose vertices are the feet of the three normals from the point $(3a, 0)$. [L.]

55. $P_1 R_1$, $P_2 R_2$ are any two parallel chords of a parabola. The chords, perpendicular to the axis of the parabola, through P_1 and P_2, cut the curve again in Q_1 and Q_2 respectively. If T_1 and T_2 are the respective poles of $Q_1 R_1$ and $Q_2 R_2$, prove that the triangles $T_1 Q_1 R_1$, $T_2 Q_2 R_2$ are equal in area. [O.]

56. Prove from geometrical considerations that the equation of a straight line in polar coordinates is of the form

$$r = a\sec(\theta-\alpha),$$

where a and α are constants, and determine the length and direction of the perpendicular on it from the origin.

By means of tables find the polar coordinates of the points in which the line with $a = 2$, $\alpha = \tan^{-1}\frac{3}{4}$ cuts the parabola

$$r = \frac{10}{1+\cos\theta}.$$ [W.]

57. Prove that the polar equation of a circle passing through the pole is of the form

$$r = d\cos(\theta-\beta),$$

and give the geometrical interpretation of the constants d, β.

The parabola $l = 2r\sin^2\frac{1}{2}\theta$ has its focus at S, and P is the point of vectorial angle α on the curve. Find the equation of the circle SPR which touches the parabola at P.

If Q is the other extremity of the focal chord through P, and the circle SQR touches the parabola at Q, prove that the equation of SR is

$$\tan(\theta-\tfrac{3}{2}\alpha) = \tan^3 \tfrac{1}{2}\alpha. \qquad \text{[J.]}$$

58. Obtain the equation of an ellipse in the form $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$.

Find the latus rectum, eccentricity, and coordinates of the foci of the ellipse $9x^2+5y^2-30y = 0$. [D.]

59. The point $(0, 3)$ is a focus and the straight line $3y = 25$ is the corresponding directrix of an ellipse of which the origin is the centre. Find the equations of the ellipse and its auxiliary and director circles.

Show that the line $3x+y = 13$ is a tangent to the ellipse, and find the coordinates of the points in which it cuts the auxiliary circle. [L.]

60. From a point P on a circle a perpendicular PN is drawn to a fixed diameter HK of the circle. If a point Q divides PN in the ratio 2 : 3, prove that the locus of Q is an ellipse, and find its eccentricity.

Prove also that the tangent to the circle at P and the tangent to the ellipse at Q intersect on HK produced. [J.]

61. Prove that the line $y = mx+\sqrt{(a^2m^2+b^2)}$ touches the ellipse $x^2/a^2+y^2/b^2 = 1$.

Find the equation of each of the two tangents drawn from the point $(27, 8)$ to the ellipse $x^2+9y^2 = 9$. [C.]

62. Show that the point

$$\left\{\frac{a(1-t^2)}{1+t^2}, \frac{2bt}{1+t^2}\right\}$$

lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1,$$

and prove that the tangents to the ellipse at the points, for which $t = t_1$ and t_2 respectively, intersect in the point

$$\left\{\frac{a(1-t_1t_2)}{1+t_1t_2}, \frac{b(t_1+t_2)}{1+t_1t_2}\right\}.$$

Any tangent to the above ellipse cuts the circle $x^2+y^2 = a^2$ in the points P and Q. Prove that the locus of the point of intersection of the remaining tangents to the ellipse from P and Q is the ellipse $$(2a^2-b^2)^2x^2+a^2b^2y^2 = a^2(2a^2-b^2)^2. \qquad \text{[O.]}$$

63. Prove that, if $a^2l^2+b^2m^2-n^2 = 0$, the line

$$lx+my+n = 0$$

touches the ellipse $x^2/a^2+y^2/b^2 = 1$ at the point

$$(-a^2l/n, -b^2m/n).$$

Find the common tangents of the two ellipses

$$x^2/a^2+y^2/b^2 = 1 \quad \text{and} \quad x^2/c^2+y^2/d^2 = 1,$$

and show that their points of contact lie on the ellipse

$$(b^2+d^2)x^2+(a^2+c^2)y^2 = a^2d^2+b^2c^2. \qquad \text{[O. and C.]}$$

64. Find the equation of the chord of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

whose mid-point is (x_1, y_1).

The chord PQ of the above ellipse always passes through the fixed point (a, b); prove that the mid-point of PQ lies on the curve

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{x}{a}+\frac{y}{b}. \qquad \text{[W.]}$$

65. Find the coordinates of any point on the ellipse

$$x^2/a^2+y^2/b^2 = 1$$

in terms of the eccentric angle of the point.

Find the eccentric angle of the point P, if the tangent at P makes equal positive intercepts on the axes, and the equation of the diameter conjugate to CP. [D.]

66. If θ and ϕ are the eccentric angles of the points P and Q on the ellipse $x^2/a^2+y^2/b^2 = 1$, prove that the equation of the chord PQ is $$\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2} = \cos\frac{\theta-\phi}{2}.$$

If PQ passes through the focus $(ae, 0)$, prove that

$$\tan\frac{\phi}{2} = -\frac{1-e}{1+e}\cot\frac{\theta}{2}.$$

Hence show that, if the chord PR passes through the focus $(-ae, 0)$, the equation of the chord QR is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta\frac{1+e^2}{1-e^2}+1 = 0. \qquad \text{[B.]}$$

67. Find the condition that the tangents to the ellipse

$$x^2/a^2+y^2/b^2 = 1$$

at the two points where the ellipse cuts the line $lx+my = 1$ should be perpendicular to each other.

Prove that every tangent to the ellipse $4x^2+9y^2 = 25$ cuts the ellipse $12x^2+18y^2 = 125$ in two points where the tangents are perpendicular to each other. [C.]

68. Find the length of a diameter POQ of an ellipse of centre O and semi-axes a, b which makes an angle $\frac{1}{4}\pi$ with the major axis.

Prove also that the tangent of the angle between OP and the normal to the ellipse at P is $(a^2-b^2)/(a^2+b^2)$. [J.]

69. Find the condition that $y = lx$, $y = mx$ should be conjugate diameters of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1.$

If a tangent to this ellipse meets the director circle

$$x^2+y^2 = a^2+b^2$$

at the points P, Q, prove that, if O is the centre, OP, OQ lie along conjugate diameters of the ellipse. [B.]

70. Find the focal distances of the point $(a\cos\phi, b\sin\phi)$ on the ellipse $b^2x^2+a^2y^2 = a^2b^2$.

If P and Q are extremities of two conjugate diameters and S is a focus, prove that

$$PQ^2-(SP-SQ)^2 = 2b^2. \qquad \text{[L.]}$$

71. Prove that the tangents to the ellipse $x^2/a^2+y^2/b^2 = 1$, at the points whose eccentric angles are $\alpha\pm\beta$, intersect at the point $a\cos\alpha\sec\beta$, $b\sin\alpha\sec\beta$.

If p_1, p_2, p_3 are respectively the lengths of the perpendiculars drawn to a variable tangent to the ellipse from two fixed points P and Q on the ellipse and from the point of intersection of the tangents at P and Q, prove that the ratio p_1p_2/p_3^2 is constant. [O. and C.]

72. Find the equation of the pair of lines joining the point (X, Y) to the points in which the line $lx+my+n = 0$ cuts the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

Prove that, if the diameter PP' of this ellipse subtends an angle λ at the point Q on the curve, then

$$4a^2b^2\cot^2\lambda = (a^2-b^2)^2\sin^2(\alpha+\beta),$$

where α, β are the eccentric angles of P, Q. [C.]

73. Define the polar of a point with respect to a conic, and find the equation of the polar of the point (x_1, y_1) with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

Find also the coordinates of the pole of the line

$$lx+my+n = 0$$

with respect to the same ellipse, and prove that if the line moves so that it always touches the circle

$$x^2+y^2 = 1,$$

its pole describes another ellipse. [W.]

74. Find the equations of the tangent and the normal to the ellipse $x^2/a^2+y^2/b^2 = 1$ at the point $(a\cos\phi, b\sin\phi)$.

Show that four normals may be drawn to an ellipse from a given point (h, k). If the eccentric angles of the feet of such normals are ϕ_1, ϕ_2, ϕ_3, ϕ_4, show that

$$\sum \cot\tfrac{1}{2}\phi_1 \cot\tfrac{1}{2}\phi_2 \cot\tfrac{1}{2}\phi_3 = 8$$

if (h, k) lies on the line

$$ax-4by = b^2-a^2.$$ [L.]

75. Prove that a chord of an ellipse is parallel to a fixed direction if the sum of the eccentric angles of its extremities is constant.

A chord of an ellipse, centre C, drawn in a fixed direction meets the major axis of the ellipse at X, and the perpendicular drawn to the chord at its middle point meets the major axis at Y. Prove that the ratio $CX : CY$ is constant. [O. and C.]

76. An ellipse and a circle intersect in four points P, Q, R, S: show that the lines PQ, RS are equally inclined to an axis of the ellipse.

Prove that at any point on the ellipse $x^2/a^2+y^2/b^2 = 1$ two circles can be drawn that touch the ellipse there and again at another point: prove also that as P moves round the ellipse the middle point of the line joining the centres of these circles moves round the ellipse whose equation is

$$a^2x^2+b^2y^2 = \tfrac{1}{4}(a^2-b^2)^2.$$ [B.]

77. If the line $lx+my = 1$ cuts the hyperbola $xy = c^2$ in two points (x_1, y_1), (x_2, y_2), prove that

$$x_1+x_2 = \frac{1}{l}, \qquad x_1 x_2 = \frac{c^2m}{l},$$

and find the corresponding values of y_1+y_2 and $y_1 y_2$.

Prove that the length of the chord of the hyperbola along the line $x+2y = 4c$ is $c\sqrt{10}$. [W.]

78. The coordinates of the vertices A, B, C of a triangle ABC are respectively (x_1, y_1), (x_2, y_2), (x_3, y_3). Prove that the angle A is acute or obtuse, according as the sign of the expression

$$(x_1-x_2)(x_1-x_3)+(y_1-y_2)(y_1-y_3)$$

is positive or negative.

Prove that any triangle whose vertices lie on the same branch of a rectangular hyperbola is obtuse-angled. [O.]

79. PQ is any chord of a rectangular hyperbola and a parallelogram $PRQT$ is drawn with PQ as diagonal and its sides parallel to the asymptotes; prove that RT passes through the centre. [D.]

80. Prove that the distances of the point $P(x', y')$ on the hyperbola $x^2/a^2-y^2/b^2 = 1$ from the foci of the hyperbola are $ex' \pm a$, where e is the eccentricity.

If the normal at P meets the conjugate axis of the hyperbola at G, prove that the square of the distance of G from either focus is in a constant ratio to the rectangle contained by the focal distances of P. [O. and C.]

81. P, Q, R are three points in this order on one branch of a rectangular hyperbola; PP', QQ' are diameters. Prove that RP, RP' are parallel to conjugate diameters and that the angles QRP, $Q'RP'$ are equal. [B.]

82. Give a careful definition of the asymptotes of a hyperbola and deduce from your definition the equation of the asymptotes of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2} = 1$.

A straight line through the centre C of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2} = 1$ meets the curve at P, and also meets at T and T' lines drawn through the vertex $(a, 0)$ parallel to the asymptotes of the curve. Prove that $CP^2 = CT.CT'$. [J.]

83. The foci of an ellipse of eccentricity e are S and S'. A hyperbola is constructed with S, S' as foci and passing through the ends of the latera recta of the ellipse; prove that its eccentricity is e^{-1}, and that it cuts the ellipse at right angles. Prove that its asymptotes intercept on the latus rectum a length equal to the minor axis of the ellipse. [D.]

84. If the tangent at any point of a hyperbola meets the asymptotes in P, Q, prove that PQ is bisected at the point of contact.

If the normal at the same point of the curve cuts the axes in L, M, prove that the four points P, Q, L, M lie on a circle which passes through the centre of the hyperbola. [L.]

85. Obtain the equation of the chord joining the two points $(c\alpha, c\alpha^{-1})$, $(c\beta, c\beta^{-1})$ on the rectangular hyperbola $xy = c^2$.

Prove that, if P, Q, R are three points on this curve $xy = c^2$, the orthocentre of the triangle PQR also lies on the curve. State the limiting form of this theorem when P and Q approach each other. [B.]

86. Obtain the coordinates of the point of intersection of the tangents to the hyperbola $xy = a^2$ at the points

$$\left(at_1, \frac{a}{t_1}\right), \quad \left(at_2, \frac{a}{t_2}\right).$$

Pairs of tangents to the hyperbola $xy = a^2$ are drawn from points on the parabola $y^2 = 4ax$. Prove that the middle points of the chords of contact lie on the parabola $4x^2 = ay$. [O.]

87. AB is a chord of the rectangular hyperbola $xy = c^2$. Prove that the circle on AB as diameter cuts the hyperbola again at points which are the ends of a diameter of the hyperbola, and that these points are real, only if the points A, B lie on different branches of the hyperbola. [C.]

88. Verify that the equation

$$2x^2 - 3xy - 2y^2 - 3x + y = 0$$

can be written in the form

$$(2x + y - 1)(x - 2y - 1) = 1,$$

and hence show that it represents a rectangular hyperbola, determine its asymptotes and centre, and write down the equations of the axes.

Draw a diagram roughly to scale, and on it mark the approximate positions of the foci. [W.]

89. Find the equation of the normal to the hyperbola

$$b^2x^2-a^2y^2 = a^2b^2$$

at the point (x', y').

The normal at a point P of this hyperbola meets the transverse axis of the hyperbola at G, and GQ is drawn at right angles to this axis and equal to GP. Find the equation of the locus of Q.

[O. and C.]

90. Prove that, if the asymptotes of a hyperbola are inclined at an angle 2θ, then the eccentricity $= \sec\theta$.

If S and S' are the foci of a rectangular hyperbola, and a tangent at P meets the asymptotes in A and B, find the angles ASB, $AS'B$, and prove that $SAS'B$ is a cyclic quadrilateral. [D.]

91. Find the condition that the line

$$px+qy = 1$$

should touch the hyperbola

$$b^2x^2-a^2y^2 = a^2b^2.$$

A circle through the foci of this hyperbola meets the asymptotes at P, P'; Q, Q' respectively, where P and Q are on the same side of the axis of x. Prove that PQ' and $P'Q$ touch the hyperbola.

[B.]

92. Find the equation (i) of the parabola, (ii) of the rectangular hyperbola which touches the x-axis at the origin and which passes through the points $(0, c)$, (λ, c).

A rectangular hyperbola touches the x-axis at the origin, and passes through the point $(0, c)$. If either asymptote of the hyperbola cuts the x-axis at $(h, 0)$, prove that $4h^2 \leqslant c^2$. [J.]

93. A straight line with gradient $\tan\theta$ meets the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$, centre O, in the points P, Q. Show that if (h, k) is the mid-point of PQ, $\tan\theta = \frac{b^2h}{a^2k}$.

If PQ meets the asymptotes in R, S, show that the area of the triangle ORS is $(b^2h^2-a^2k^2)/ab$.

The circle on RS as diameter meets the asymptotes again in R', S'; show that $\dfrac{\Delta OR'S'}{\Delta ORS} = \left(\dfrac{a^2-b^2}{a^2+b^2}\right)^2$. [L.]

94. Prove that the straight line

$$\left(\frac{x}{a}+\frac{y}{b}\right)+\left(\frac{x}{a}-\frac{y}{b}\right)t_1 t_2-(t_1+t_2) = 0$$

cuts the hyperbola $\dfrac{\frac{x}{a}+\frac{y}{b}}{t^2} = \dfrac{\frac{x}{a}-\frac{y}{b}}{1} = \dfrac{1}{t}$, where t is a variable parameter, in the points at which t has the values t_1 and t_2 respectively, and hence prove that if m is the gradient of the chord through these points,

$$t_1 t_2 = \frac{am+b}{am-b}.$$

P_1, P_2, P_3, P_4 are variable points on a hyperbola, such that the gradients of the chords P_1P_2, P_2P_3, P_3P_4 are constant. Prove that the gradient of P_1P_4 is constant. [O.]

95. *Find in its simplest form the condition that the four points* with coordinates $(kt, k/t)$, where the parameter t has the values a, b, c, d, should lie on a circle.

A, B, C, D are four points on a rectangular hyperbola and are not concyclic; if the circles BCD, CAD, ABD, ABC meet the rectangular hyperbola again in the points α, β, γ, δ respectively, prove that the middle points of the chords $A\alpha$, $B\beta$, $C\gamma$, $D\delta$ lie on another rectangular hyperbola with the same asymptotes. [C.]

96. Prove that the envelope of a chord of a parabola which subtends an angle of 60° at the focus is an ellipse with the same focus and same directrix and of eccentricity $\sqrt{3}/2$. [W.]

97. Find the coordinates of the centre, the lengths and equations of the axes of the conic

$$5x^2+8xy+5y^2 = 12x+6y.$$

Find also the equation of a directrix. [L.]

98. Find the condition that the straight lines

$$lx+my+n = 0 \quad \text{and} \quad l'x+m'y+n' = 0$$

should be at right angles, the angle between the axes of coordinates being ω.

5. $(1\frac{4}{5}, 2\frac{2}{5})$.
6. $(3, -1)$; $(4\frac{1}{2}, 0)$. $(3\frac{3}{4}, -\frac{1}{2})$.
7. 11.
8. Both $(3, -\frac{3}{4})$.
9. $4\sqrt{17}$.
10. (i) It bisects the line.
    (ii) It divides the line in the ratio 5 : 3.
11. $(\frac{1}{3}, 2\frac{2}{3})$.
12. $(\frac{2}{3}, 3\frac{1}{3})$.
13. $(8, 14)$.
14. $\sqrt{\{r_1^2+r_2^2-2r_1 r_2\cos(\theta_1-\theta_2)\}}$.
15. $\dfrac{45\sqrt{3}}{4}$.

EXAMPLES 3 (*p.* 28)

1. No; Yes; No; No; Yes.
4. $x\sqrt{3}-y = \pm 6$.
5. $x\sqrt{3}-y = 2$.
6. $x\sqrt{3}+y = \sqrt{3}$.
7. $x\sqrt{3}+y = \frac{5}{2}\sqrt{3}$.
8. $3a-2b = ab$. ✓
9. (1) $-\frac{5}{7}$; $1\frac{4}{5}$, $1\frac{2}{7}$. (2) 1; 6, -6. (3) $\frac{4}{5}$; $2\frac{1}{4}$, $-1\frac{4}{5}$.
11. $4\frac{1}{2}$.
12. $x\cos 112^\circ\ 37'+y\sin 112^\circ\ 37' = 1$.
13. $1\frac{2}{3}$.
14. $-1\frac{1}{3}$.
15. $\frac{3}{5}$.
17. The line at infinity.
18. $3x+2y = 12$.

EXAMPLES 4 (*p.* 38)

1. $2x+3y = 0$; $x+5y+23 = 0$; $2x-y = a$.
2. $x-\sqrt{3}y = 3+4\sqrt{3}$.
3. $ax+by = a^2+b^2$.
4. $\dfrac{m'-m}{lm'-l'm}, \dfrac{l-l'}{lm'-l'm}$.

5. (1) $(3\frac{9}{11}, -\frac{1}{11})$. (2) $\left(-\frac{c(a-b)}{a^2+b^2}, -\frac{c(a+b)}{a^2+b^2}\right)$.

(3) $\left(\frac{p'\sin\alpha - p\sin\alpha'}{\sin(\alpha-\alpha')}, \frac{p\cos\alpha' - p'\cos\alpha}{\sin(\alpha-\alpha')}\right)$.

6. $x+y = 2$.

7. $x = 4y$.

8. $b^2x-a^2y = ab(b-a)$.

11. (1) $2\sqrt{2}$. (2) $5\frac{1}{5}$. (3) $1\frac{7}{13}$. (4) $\sqrt{3}+\frac{1}{2}$.

12. $4x-5y = 2$; $4x+5y = 22$.

13. $\frac{c-d}{\sqrt{(1+m^2)}}$.

14. $(0,4)$, $(-1,-2)$ on the same side.
$(4,1)$, $(5,-2)$ on the opposite side.
$(2,2)$ is on the line.

15. $(1, 4\frac{1}{2})$.

16. $2x(b+d-b'-d')-2y(a+c-a'-c') = (b+d)(a'+c')-(b'+d')(a+c)$.

17. $x = y$; $x+3y = 4$; $2x+y = 3$. In-centre is $(1,1)$.

18. $x(b+b') = y(a+a')$; $x(2b-b')-y(2a-a') = a'b-ab'$;
$x(2b'-b)-y(2a'-a) = ab'-a'b$.

20. $(-2,3)$.

21. $2x+3y = 17$.

22. $(l-l')x+(m-m')y = 0$.

23. $23x-17y = 103$.

24. (1) $x\cos\frac{\alpha+\beta}{2}+y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}$.

(2) $x-\frac{1}{2}(t+t')y+att' = 0$.

(3) $\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2} = \cos\frac{\theta-\phi}{2}$.

(4) $x+y\mu\mu' = c(\mu+\mu')$.

25. (1) a; (2) $\frac{at^2}{\sqrt{(1+t^2)}}$; (3) $\frac{ab}{\sqrt{(a^2\sin^2\phi+b^2\cos^2\phi)}}$; (4) $\frac{2c\mu}{\sqrt{(1+\mu^4)}}$.

26. (1) at_1t_2, $a(t_1+t_2)$. (2) $\dfrac{a\cos\dfrac{\theta+\phi}{2}}{\cos\dfrac{\theta-\phi}{2}}$, $\dfrac{b\sin\dfrac{\theta+\phi}{2}}{\cos\dfrac{\theta-\phi}{2}}$.

(3) $\dfrac{2c\mu_1\mu_2}{\mu_1+\mu_2}$, $\dfrac{2c}{\mu_1+\mu_2}$.

27. $x(cn-dm) = y(c-d)$; $x(md+nc)-y(c+d)+cd = 0$.

29. $\dfrac{\sqrt{(c^2-2cd\cos\alpha+d^2)}}{\sin\alpha}$, $\tan^{-1}\dfrac{d-c\cos\alpha}{c\sin\alpha}$; c, $\dfrac{d-c\cos\alpha}{\text{[illegible]}\sin\alpha}$.

31. $x(a\sin\alpha+b\cos\alpha) = y(a\cos\alpha-b\sin\alpha)$.
$x(b\cos\alpha-a\sin\alpha)+y(a\cos\alpha+b\sin\alpha) = ab$.

33. $\dfrac{\sin(\theta_2-\theta_1)}{r} = \dfrac{\sin(\theta-\theta_1)}{r_2}+\dfrac{\sin(\theta_2-\theta)}{r_1}$.

34. $x-8y+9 = 0$; $7x-y = 8$; $10x-3y+5 = 0$.

35. 7 : 4.

36. $\frac{1}{2}[r_2r_3\sin(\theta_3-\theta_2)+r_3r_1\sin(\theta_1-\theta_3)+r_1r_2\sin(\theta_2-\theta_1)]$.

37. $-\dfrac{l}{2\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}}$, $\dfrac{\alpha+\beta}{2}$.

39. $r\cos(\theta-4\alpha) = 3\cos\alpha\cos 2\alpha$.

40. $8\frac{1}{4}$.

EXAMPLES 5 (*p.* 55)

1. (1) 29° 45′. (2) 59° 29′. (3) 37° 14′. (4) 32° 28′.

2. (1) $7x+4y+47 = 0$. (2) $x\sin\alpha-y\cos\alpha = 3\cos\alpha-5\sin\alpha$.
(3) $3x-4y+3 = 0$.

3. -5.

4. (1) $3x+y = 2$; $x-3y = 4$.
(2) $2x-y = 4$; $2x+11y = 28$.
(3) $2x+3y = 7$.

5. θ.

6. (1) 90°. (2) 90°. (3) 90°. (4) 62° 6′. (5) 15°.

7. (1) 15. (2) 11 or $15\frac{2}{3}$. (3) 21 or $-22\frac{2}{3}$.

8. (1) $y(\sqrt{2}-1)+x(3-2\sqrt{2}) = 0$; $y(\sqrt{2}+1)-x(3+2\sqrt{2}) = 0$.
(2) $27x-99y+35 = 0$; $77x+21y+95 = 0$.
(3) $9x+27y+29 = 0$; $39x-13y-41 = 0$.

9. $77x-99y-131 = 0$; $9x+7y = 17$. The latter.

10. (1) $x^2-y^2 = 0$. (2) $4(x^2-y^2)+xy = 0$.
(3) $(a-b)(x^2-y^2) = 2cxy$.

11. (1) $(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}})$. (2) $(0,0)$. (3) $(-0{\cdot}1, 4{\cdot}3)$.

12. $(0{\cdot}7, 4{\cdot}3)$.

14. (1) $35x^2+108xy+80y^2 = 0$.
(2) $y^2-12axy-8ax^2 = 0$.
(3) $60x^2+177xy+135y^2 = 0$.

15. $(-\frac{1}{2}, 0)$.

16. $x^2(c-2mg)+y^2c+2gxy = 0$. $c = mg$.

17. $(5{\cdot}84, -2{\cdot}12)$.

18. (1) $(\frac{6}{7}, \frac{17}{7})$. (2) $(u+v+t+uvt, 2)$.

EXAMPLES 6 (*p.* 66)

2. $200x+250y = 367$.

3. $a = 3$, $b = 3$. $(-1,1)$, $(-\frac{7}{13}, \frac{17}{13})$.

5. -12, 90°.

6. $x\cos\alpha + y\sin\alpha = p \pm d$.

7. $39x+29y+10 = 0$.

8. $(\frac{3}{2}, \frac{3}{2})$.

10. $(-\frac{23}{4}, \frac{7}{4})$.

12. $\lambda^2+\lambda(a^2+b^2-h^2-k^2)-b^2h^2-a^2k^2+a^2b^2 = 0$.

14. $\left(\frac{2X-Y}{2}, \frac{3Y-X}{2}\right)$; $2(x+y) = 1$.

16. $2x+3y = 8$ or $2x-9y+8 = 0$.

18. (1) $(l'm-lm')(ax-by) = a(m-m')+b(l-l')$.
(2) $(l'm-lm')(ay+bx) = a(l-l')+b(m-m')$.

20. 36.

23. (1) $(0, 2R\cos B\cos C)$.
(2) $\left(\frac{b\cos C - c\cos B}{3}, \frac{b\sin C}{3}\right)$.
(3) $\left(\frac{b\cos C - c\cos B}{2}, \frac{b\sin^2 C - c\cos B\cos C}{2\sin^2 C}\right)$.

24. $b(x-x')^2+a(y-y')^2-2h(x-x')(y-y') = 0$.

27. $3x^2+4xy-5y^2 = 0$.

29. $(14, -21)$.

32. $r\sin(\theta-\alpha) = r_1\sin(\theta_1-\alpha)$.

EXAMPLES 7 (*p.* 83)

1. (1) Goes through $(0, -4)$. Symmetrical about the y-axis. No part of the curve is to the left of $y+4 = 0$. Goes to infinity in two directions on the right of this line. A parabola.
   (2) Symmetrical with regard to the origin and axes. A closed curve lying entirely within a rectangle formed by the lines $x = \pm 1$, $y = \pm\frac{1}{3}$. An ellipse.
   (3) Symmetrical with regard to the origin and axes. Has two branches one in the first and the other in the third quadrant. Each branch goes off to infinity in the directions of the axes. A hyperbola.
   (4) Symmetrical with regard to the point $(2, 3)$, its centre. Also with regard to the lines $x = 2$, $y = 3$. Goes off to infinity in the directions of these lines with two branches. A hyperbola.
   (5) Symmetrical with regard to the origin, which is its centre. Also with regard to the lines $3x = y$, $x = 2y$. Goes off to infinity in the directions of these lines with two branches. A hyperbola.

3. Goes through the point $(0, 2)$. Touches the y-axis and is symmetrical with regard to the line $y = 2$. No part of the curve lies to the left of the y-axis. Cuts the x-axis in the point $(\frac{4}{3}, 0)$. Goes to infinity in two directions. A parabola.

5. (1) $9x^2+4y^2 = 19$. (2) $3xy = 8$. (3) $5x^2+4xy-3y^2-2y = 1$.
   (4) $x^2+y^2 = 36$.

7. 53° 3′ or 143° 3′.

8. (1) $(-\frac{24}{25}, \frac{18}{25})$. (2) $(-4, 1)$. (3) $(\frac{15}{11}\sec\theta, \frac{14}{11}\operatorname{cosec}\theta)$.

9. $2xy+a^2 = 0$.

10. $y = m(x+a)+\dfrac{a}{m}$.

EXAMPLES 8 (*p.* 105)

1. (1) $x^2+y^2+6x-4y-12 = 0$.
   (2) $x^2+y^2-2x+8y-1 = 0$.
   (3) $x^2+y^2-2ax-2by+a^2+b^2-r^2 = 0$.

2. (1) $(4, 0)$; 3. (2) $(\frac{3}{2}, -\frac{5}{2})$; $\frac{1}{2}\sqrt{38}$. (3) $\left(-\frac{a}{2}, -\frac{b}{2}\right)$; $\frac{1}{2}\sqrt{(a^2+b^2-4c)}$.
   (4) $(a, -b)$; b.

3. (i) $11(x^2+y^2)+9x-137y = 0$.
   (ii) $4(x^2+y^2)+2x-13y-42 = 0$.
   (iii) $(a-b)(x^2+y^2)-(a^2+b^2)(x+y)+2b(a^2+b^2) = 0$.
   (iv) $c(x^2+y^2)-c(a+b)x-(ab+c^2)y+abc = 0$.

4. $x^2+y^2-6x+2y-15 = 0$.

6. $(5, -9)$.

7. $x+2y+6 = 0$.

8. $x^2+y^2-4x-6y-48 = 0$.

9. $x^2+y^2-8x+6y = 0$.

10. $x^2+y^2+10x+5y+30 = 0$.

12. 45°.

13. (1) $4x-3y+25 = 0$; $3x+4y = 0$.
    (2) $x+\sqrt{3}y = 8$; $\sqrt{3}x = y$.
    (3) $x\cos\alpha + y\sin\alpha = 7$; $x\tan\alpha = y$.
    (4) $4x-3y+19 = 0$; $3x+4y = 17$.
    (5) $2x = y$; $x+2y = 10$.
    (6) $17x+5y+61 = 0$; $5x-17y = 19$.

15. $6\pm 2\sqrt{10}$.

17. $c = \pm 15$.

18. $c = \pm 8\sqrt{26}-7$.

20. $x^2+y^2+20x+20y+100 = 0$; $x^2+y^2+4x+4y+4 = 0$.

21. (1) $x^2+y^2 = 4$. (2) $x^2+y^2 = 36$. (3) $(l^2+m^2)(x^2+y^2) = 1$.

22. $x^2+y^2-3x-10y+25 = 0$.

23. $x^2+y^2-8\sqrt{2}x-8(2-\sqrt{2})y+32 = 0$.

27. $(x-h)\cos\alpha+(y-k)\sin\alpha = a$.

28. $y+\sqrt{3}x = 7-5\sqrt{3}$; $y+\sqrt{3}x+5\sqrt{3}+1 = 0$.

32. $(3, 4)$; $(-\frac{7}{5}, -\frac{24}{5})$.

35. $x\cos 30° + y\sin 30° = a\cos 15°$.

36. $3x-2y = 13$.

37. $x\cos 82\frac{1}{2}° - y\sin 82\frac{1}{2}° + 3\cos 22\frac{1}{2}° = 0$.

39. $21(x^2+y^2)-82x+18y-251 = 0$.

40. $23x^2-2y^2+5xy = 0$.

41. $b^2(a^2-c^2)x^2+a^2(b^2-c^2)y^2-2abc^2xy = 0$.

42. $x^2-y^2 = a^2-b^2$.

43. $x^2-2yh = l^2-h^2$, fixed point being $(0, h)$ and length of line $2l$.

44. 48° 11′; 131° 49′.

45. $a^2(x^2+y^2) = 4x^2y^2$.

46. $x\cos\frac{\alpha}{2}+y\sin\frac{\alpha}{2} = a$, where $\alpha = \sin^{-1}\frac{2a}{b}$.

49. $\{cy\pm b(x+a)\}^2+\{by\mp c(x-a)\}^2 = (x^2+y^2-a^2)^2$.

51. $xh+yk+g(x+h)+f(y+k)+c = 0$.

52. $gx+fy+c = 0$.

53. $(\frac{109}{22}, -\frac{67}{11})$.

54. $bx+(a-2b)y = a^2+2ab-b^2-c$.

57. $xh+yk = a^2$.

60. $x^4+(x^2-a^2)y^2+2a^2x^2+a^4 = 0$.

63. $3(x^2+y^2)-16x+8\sqrt{5}y+36 = 0$, and
$3(x^2+y^2)-32x+16\sqrt{5}y+180 = 0$.

64. $x^2+y^2-ax-by = \pm(ay+bx-ab)$.

67. $\frac{a}{2}(4-t_1t_2-t_2t_3-t_3t_1)$, $\frac{a}{4}t_1t_2t_3$.

68. $(x-a)^2+(y-b)^2 = r^2$.

EXAMPLES 9 (*p.* 131)

1. (1) $40x^2-84xy+27y^2-108x-126y-765 = 0$; $2\sqrt{19}$.
(2) $x^2(\lambda^2+\delta^2-\kappa^2)+2\kappa\lambda xy+y^2\delta^2-2\kappa\lambda^2x-$
$-2\delta^2\lambda y+\lambda^2\delta^2 = 0$; $\sqrt{(\lambda^2+\delta^2)}$.
(3) $15x^2+70xy+39y^2-410x-274y+71 = 0$; 8.
(4) $57x^2-176xy+1936x+330y-10527 = 0$; 11.

2. $\frac{1}{2}$.

5. $b(x^2+y^2)-b^2x-(a^2+b^2)y+a^2b = 0$.

6. Their radical axis.

7. $39(x^2+y^2)-61x+15y-104 = 0$.

8. 12.

10. $2(x^2+y^2)+22x+10y-43 = 0$.

13. It lies on the common chord and within the two circles.

14. $2x-3y+18 = 0$; $(-\frac{33}{26}, \frac{67}{13})$.

16. $6(x^2+y^2)-148x-71y+166 = 0$.

17. $(\pm 3, 0)$.

18. $3(x^2+y^2)+8x-3 = 0$; $x^2+y^2 = 1$.

19. $3(x^2+y^2)+23x-31y = 0$.

20. $8(x^2+y^2)+5(3\pm\sqrt{41})\delta x+8\delta^2 = 0$.

23. $4(x^2+y^2) = l^2+4\delta^2$.

24. $x^2-y^2+2cxy-\delta^2 = 0$, where c is a constant.

27. $(\kappa+\lambda)(x^2+y^2)-2\kappa\lambda(x+y)-(\kappa-\lambda)\delta^2 = 0$.

29. (1) $(\frac{4}{3}, -\frac{16}{9})$; $(-12, 16)$.

(2) $\dfrac{\pm\kappa\sqrt{(\lambda^2+\delta^2)}}{\sqrt{(\kappa^2-\delta^2)}\pm\sqrt{(\lambda^2+\delta^2)}}, \dfrac{\lambda\sqrt{(\kappa^2-\delta^2)}}{\sqrt{(\kappa^2-\delta^2)}\pm\sqrt{(\lambda^2+\delta^2)}}$.

30. $11x^2-60xy+200x+240y-976 = 0$, and
$27x^2-60xy+16y^2+120x+144y-976 = 0$.

33. $5(x^2+y^2)+12x-3 = 0$.

38. $r\cos(\theta-\alpha-\beta) = 2a\cos\alpha\cos\beta$.

40. $(\sqrt{2}a, 45°)$; $(-\sqrt{2}a, 135°)$.

41. $\sqrt{(a^2+b^2)}$.

42. $c(2a+b^2c) = 1$.

45. (1) $r = A\cos(\theta-B)$, where $A\cos B = a\cos\alpha+b\cos\beta$, and $A\sin B = a\sin\alpha+b\sin\beta$.

(2) $r(\lambda+1) = A\cos(\theta-B)$, where $A\cos B = a\cos\alpha+\lambda b\cos\beta$, and $A\sin B = a\sin\alpha+\lambda b\sin\beta$.

46. A circle whose radius is to that of the given circle as $k : k+l$.

47. $x = y$; $\dfrac{1}{\sqrt{2}}(a-b)$.

51. (1) $(x-h)^2+(y-k)^2+x^2+y^2 = a^2$.

(2) $(h^2+k^2-a^2)(x^2+y^2)-2a^2hx-2a^2ky+2a^4 = 0$.

52. $\left(\dfrac{\delta^2}{\kappa}, 0\right)$.

EXAMPLES 10 (*p.* 162)

1. (1) $(4, 0)$; $(7, 0)$; 12; $x = 1$.

(2) $(-c, 0)$; $(a-c, 0)$; $4a$; $x+c+a = 0$.

(3) $(0, -c)$; $(0, a-c)$; $4a$; $y+c+a = 0$.

(4) $(-3, \frac{5}{2})$; $(-3, 3)$; 2; $y = 2$.

(5) $(7, 0)$; $(6\frac{1}{4}, 0)$; 3; $4x = 31$.

(6) $(-2c, b)$; $(-2c+\dfrac{3a}{4}, b)$; $3a$; $4x+8c+3a = 0$.

(7) $(2, 2)$; $(1\frac{1}{2}, 2)$; 2; $2x = 5$.

(8) $(2\frac{1}{3}, 2)$; $(\frac{5}{6}, 2)$; 6; $6x = 23$.
(9) $(10\frac{1}{3}, -2)$; $(9\frac{7}{12}, -2)$; 3; $12x = 133$.
(10) $(-3, -\frac{32}{7})$; $(-3, -\frac{79}{28})$; 7; $28y+177 = 0$.
(11) $(6\frac{1}{3}, -1)$; $(6\frac{7}{48}, -1)$; $\frac{3}{4}$; $48x-313 = 0$.
(12) $(\frac{3}{25}, \frac{4}{25})$; $(0, 0)$; $\frac{4}{5}$; $3x+4y-2 = 0$.

2. (1) $x^2-2ax+2ay+2a^2 = 0$.
(2) $16x^2+24xy+9y^2-2x-14y-14 = 0$.
(3) $144x^2+120xy+25y^2-416x-1638y+1521 = 0$.
(4) $m^2x^2-2lmxy+l^2y^2-2x(pl^2+pm^2-l)-$
$-2y(ql^2+qm^2-m)+(l^2+m^2)(p^2+q^2)-1 = 0$.

3. $16x^2+24xy+9y^2-40x-230y+225 = 0$.

4. 0; 1; -1; -2; $\dfrac{1}{\sqrt{2}}$.

5. (1) $a(y-1)^2 = c^2(x-1)$. (2) $(ax-by)^2 = (bc-a^2)(cx-ay)$.

6. $(2, 0)$; $(1, 0)$; $(0, -4)$; $(0, -2)$.

7. $gx^2 = 2u^2 \sin\alpha\cos\alpha x - 2u^2\cos^2\alpha y$.

8. $(x+3)^2 = 8y+28$.

9. $(y-k)^2 = 4c(x-h)$.

10. (1) $x+8y+16 = 0$; $8x-y = 132$.
(2) $x+y+3 = 0$; $x-y = 9$.
(3) $x-y-3 = 0$; $x+y = 9$.
(4) $2y-x = 5$; $2x+y = 20$.
(5) $2x-y = 16$; $x+2y = 18$.
(6) $2yy'+ax+by-y'^2 = 0$, $(2y'+b)(x-x') = a(y-y')$.

11. $x-y+a = 0$; $x+y+a = 0$. $x+y = 3a$; $x-y = 3a$.

12. (1) $c = \dfrac{9}{2}a$. (2) $c = -a$.

17. $(a\cot^2\theta, 2a\cot\theta)$.

18. $\sqrt{3}x-y = 5a\sqrt{3}$.

20. $2x-y-4 = 0$; $2x+4y+1 = 0$.

23. $(125, 50)$.

25. $a(t_1^2+t_1t_2+t_2^2+2)$; $-at_1t_2(t_1+t_2)$.

27. $b(t_1+t_2)^2 = 4at_1t_2$.

30. $\left(a\cot^2\dfrac{\theta}{2}, 2a\cot\dfrac{\theta}{2}\right)$; $\left(a\tan^2\dfrac{\theta}{2}, -2a\tan\dfrac{\theta}{2}\right)$.

33. (1) $y = a\kappa$. (2) $x = a\kappa$. (3) $y^2-2ax = a^2\kappa$. (4) $y = \kappa x$.

35. $x^2\tan^2\alpha-y^2+2ax(\tan^2\alpha+2)+a^2\tan^2\alpha = 0$.

43. $(-p\sec\alpha, -2a\tan\alpha)$.

56. $l^2 = y^2-4ax$, where l is the intercept.

59. $xy+y(2a-h) = 2ak$.

Examples 11 (*p.* 184)

2. $y^2 = 2a(x-a)$.

3. $(y^2-4ax)(y^2+4a^2)+a^2c^2 = 0$.

4. $y^4+8a^4 = 2ay^2(x-2a)$.

5. $y^2-ky = 2a(x-h)$.

9. $\left(\dfrac{l^2c^2h}{n^2}, -\dfrac{lck}{n}\right)$.

10. $(4ah-k^2)\operatorname{cosec}^2\alpha$.

13. $y^2-4ax+k^2\sin^2\alpha = 0$.

14. $2x^2+3xy+2y^2+x+6y+8 = 0$.
    $2x^2+5xy-y^2+7x-16y-31 = 0$.

15. $ly+2am = 0$.

16. $x+ty+a(t^2+4) = 0$.

23. $y^2(2x+a) = a(3x+a)^2$.

26. $2y^2 = a(x-a)$.

28. $y^2 = a(x-a)$.

29. $a\sqrt{(1+t^2)}$.

40. $ay^2 = h^2(x-2a-h)$.

42. $y^2 = 16a(x-6a)$.

46. $(x-5a)^2(2x-a) = 27ay^2$.

51. $2x^2+y^2-2x(h-2a)-3ky+4ah = 0$.

52. $x^2+y^2-10ax+9a^2 = 0$; $x^2+y^2-2ax+a^2 = 0$.

57. $\left(\dfrac{am^2}{l^2}, \dfrac{2am}{l}\right)$; $\left(\dfrac{4l^2+4m^2-ln-am^2}{2l^2}, \dfrac{lmn-am^3}{2l^2}\right)$.

Examples 12 (*p.* 226)

1. $\pm\dfrac{3\sqrt{21}}{5}$; $\pm\dfrac{12}{5}$; $\pm\dfrac{3\sqrt{91}}{10}$.

2. $\dfrac{\sqrt{7}}{4}$.

3. 2.

4. $11\sqrt{2}$.

5. $\dfrac{x^2}{64}+\dfrac{y^2}{48} = 1$.

6. $8x^2+9y^2 = 64$.

7. $25x^2+81y^2 = 2025$.

8. (1) $\frac{4}{5}$; $4x\pm 25 = 0$; $3\frac{3}{5}$.
   (2) $\dfrac{\sqrt{3}}{2}$; $x = 2\pm 4\sqrt{3}$; 3.
   (3) $\dfrac{1}{\sqrt{3}}$; $x-5 = 0$, $x+1 = 0$; $\dfrac{4\sqrt{3}}{3}$.

9. $x^2+4y^2 = 64$.

10. $651x^2+120xy+532y^2-2074x-4056y+8788 = 0$.

11. $(0,0)$, $\left(-\frac{1}{3}, -\dfrac{1}{\sqrt{3}}\right)$; 1; $x+\sqrt{3}y = 2$; $3x+3\sqrt{3}y+10 = 0$.

12. $\dfrac{1}{\sqrt{2}}$.

14. 45°, 135°, 225°, 315°.

16. $16x^2+y^2 = l^2$, where $PR = l$.

18. $(4, 4\pm 5\sqrt{3})$.

19. $9x^2+25y^2+90x = 0$.

20. $16x^2+25y^2-160x-200y+400 = 0$.

23. $2x+2\sqrt{2}y = \sqrt{3}+1$.

24. (1) $3x-10\sqrt{2}y+45 = 0$; $30\sqrt{2}x+9y+32\sqrt{2} = 0$.
    (2) $7x\cos\alpha + 8y\sin\alpha = 56$; $8x\sin\alpha - 7y\cos\alpha = 15\sin\alpha\cos\alpha$.
    (3) $x+4\sqrt{2}y+16\sqrt{2}-21 = 0$; $4\sqrt{2}x-y = 18\sqrt{2}+4$.

25. ± 7.

26. $a = \pm 2\sqrt{3}$, $b = \pm\sqrt{33}$.

27. $y = \pm x\pm\sqrt{13}$.

28. $a+b$.

29. $2(x^2+y^2) = 15$.

30. (1) $(\frac{4}{5}, \frac{4}{5})$. (2) $(2, -\frac{3}{4})$. (3) $(0, -\frac{4}{3})$.

31. $NT = a\sin^2\phi\sec\phi$; $NG = a\cos\phi(1-e^2)$.

33. $b^2(x-h)^2+a^2(y-k)^2 = a^2b^2$.

35. $(\pm\frac{2}{3}\sqrt{\frac{11}{3}}, \pm\frac{4}{5})$; 8° $4\frac{1}{2}'$.

38. $\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2} = 1$.

39. $\left(\pm\frac{a^2}{\sqrt{(a^2+b^2)}}, \pm\frac{b^2}{\sqrt{(a^2+b^2)}}\right)$.

47. $\{(a+b)\cos\phi, (a+b)\sin\phi\}$.

48. $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$.

53. (1) (a^2l, b^2m). (2) $\left(\frac{a^2\cos\alpha}{p}, \frac{b^2\sin\alpha}{p}\right)$. (3) (a, b).

(4) $\left(\frac{a}{e^2\cos\phi}, -\frac{b^3}{a^2e^2\sin\phi}\right)$.

55. $\frac{x^2}{a^2}+\frac{y^2}{b^2} = \left(\frac{a^2-b^2}{a^2+b^2}\right)^2$.

63. $a^2(x+a)^2+b^2y(y-2a) = 0$.

66. $4a^2b^2x^2+4y^2(2a^2-b^2)^2 = b^2(2a^2-b^2)^2$.

72. $r^2 = a^2\cos^2\theta+b^2\sin^2\theta$; $a^2x^2+b^2y^2 = (x^2+y^2)^2$.

73. $\frac{l}{r} = -e\cos\theta$; $\frac{l}{r} = e\cos\theta\,\frac{1-e^2}{1+e^2}$.

74. $\frac{l}{r} = \cos\lambda - e\cos\theta$.

75. $\frac{l}{r} = \cos(\theta-\alpha)+\sin(\theta-\alpha)-e\cos\theta$.

EXAMPLES 13 (*p*. 262)

2. (1) $9y+4x = 0$. (2) $9y-4x = 0$. (3) $9y+2x = 0$.

8. $\left(\pm\frac{ab}{\sqrt{(b^2+a^2m^2)}}, \pm\frac{mab}{\sqrt{(b^2+a^2m^2)}}\right)$;

$\left(\mp\frac{mab}{\sqrt{(a^2+b^2m^2)}}, \pm\frac{ab}{\sqrt{(a^2+b^2m^2)}}\right)$.

9. $b^2x^2+a^2y^2 = \frac{1}{2}a^2b^2$.

10. $b^2x^2+a^2y^2 = 2a^2b^2$.

11. $2(x^2+y^2)^2 = a^2x^2+b^2y^2$.

18. $ae^2\sin\phi\cos\phi(\sin\phi - \cos\phi)$; $-\frac{a^2e^2}{b}\sin\phi\cos\phi(\sin\phi + \cos\phi)$.

17. $b^2x^2+a^2y^2 = 13a^2b^2$.

22. $x^2(b-ka)^2+y^2(a-kb)^2 = (a-kb)^2(b-ka)^2$.

26. $b^2x^2+a^2y^2 = 4x^2y^2$.

27. $4b^2x^2+a^2(2y-1)^2 = a^2b^2$.

28. $a^2b^2(b^4x^2+a^4y^2) = (b^2x^2+a^2y^2)^2(a^2+b^2)$.

29. $l^2a^2b^2(b^2x^2+a^2y^2)+(a^4x^2+b^4y^2)(b^2x^2+a^2y^2-a^2b^2) = 0$.

30. $\tan^{-1}\dfrac{1}{e^2}$.

31. $a^2x^2+b^2y^2 = a^2(a^2-b^2)$.

33. $\tan\theta = \dfrac{2\sqrt{(b^2h^2+a^2k^2-a^2b^2)}}{h^2+k^2-a^2-b^2}$

34. $\tan^{-1}\dfrac{2a\sin\theta}{b^2}\sqrt{(a^2-b^2)}$.

35. $a^4b^4(x^2+y^2) = (a^2+b^2)(b^2x^2+a^2y^2)^2$.

36. $\dfrac{16\sqrt{2}}{3}$.

38. $(a^2-b^2)(b^2x^2\cos^2\alpha-a^2y^2\sin^2\alpha)$
$= a^2b^2(a^2+b^2)-2ab(b^3x\cos\alpha+a^3y\sin\alpha)$.

41. $y^4(a^2-l^2)+b^2y^2(x^2-a^2+2l^2)-b^4l^2 = 0$.

43. $b^2x^2+a^2y^2 = a^2b^2\cos^2\alpha$.

44. $b^2x^2+a^2y^2 = \frac{1}{2}a^2b^2$.

45. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = \dfrac{a^2b^2}{a^2+b^2}$.

49. $4x^2(a^2-\lambda^2)+4a^2y^2+4\lambda x(a^2-\lambda^2) = (a^2-\lambda^2)^2$, where $(\lambda, 0)$ is the fixed point, and $x^2+y^2 = a^2$ the fixed circle.

51. $\dfrac{4n-3}{2}\pi-3\phi$.

EXAMPLES 14 (*p.* 302)

1. $\pm 2\sqrt{3}$; $\pm 4\sqrt{2}$; $\pm 2\sqrt{15}$.

2. (1) $\theta = 120°$. (2) $\theta = -45°$. (3) $\mu = \frac{4}{3}$.

3. $\dfrac{2\sqrt{3}}{3}$.

4. $\sqrt{3}$.

5. Vertices: (c, c), $(-c, -c)$. Foci: $(c\sqrt{2}, c\sqrt{2})$, $(-c\sqrt{2}, -c\sqrt{2})$. Directrices: $x+y = \pm\dfrac{c}{\sqrt{2}}$.

6. (1) $(0,0)$; $(\pm\sqrt{74}, 0)$; $\dfrac{\sqrt{74}}{7}$; 14 and 10.

   (2) $(2, -3)$; $(2\pm\sqrt{73}, -3)$; $\dfrac{\sqrt{73}}{8}$; 16 and 6.

   (3) $(\frac{2}{5}, -\frac{2}{3})$; $(\frac{2}{5}\pm\sqrt{34}, -\frac{2}{3})$; $\dfrac{\sqrt{34}}{3}$; 10 and 6.

7. (1) $5x+7y = 0$; $5x-7y = 0$.
   (2) $3x+8y+18 = 0$; $3x-8y = 30$.
   (3) $5x+3y = 0$; $5x-3y = 4$.

9. (1) $7x^2+24xy+6x-8y-1 = 0$.
   (2) $451x^2+1080xy-620y^2+1712x-3568y+3236 = 0$.
   (3) $11x^2-96xy+39y^2 = 48$.

10. (1) $8x-\sqrt{15}y = 49$; $\sqrt{15}x+8y = 16\sqrt{15}$.
    (2) $7x-4\sqrt{3}y = 28$; $4\sqrt{3}x+7y = 113\sqrt{3}$.
    (3) $12x-7\sqrt{3}y = 6+14\sqrt{3}$; $7\sqrt{3}x+12y = 149\sqrt{3}-24$.

11. $5x+2\sqrt{3}y = 68$.

12. $(1, -\sqrt{3})$.

14. $\left(\dfrac{2a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}\right)$.

15. $c = \pm 12$.

16. $\left(\dfrac{14\sqrt{3}}{3}, \dfrac{2\sqrt{3}}{3}\right)$.

17. $9x = 64y$.

41. $(\frac{2}{3}, \frac{1}{4})$.

42. $c = 15\sqrt{745}$.

43. $x^2+y^2+14xy-16x-16y+16 = 0$; $\tan^{-1}(-7\pm 4\sqrt{3})$.

44. $p^2(b^4x^2+a^4y^2) = a^4b^4$.

45. $ex+a = 0$.

46. $b^4x^2+a^4y^2 = a^2b^4$.

47. Hyperbola equation $xy = c^2$: $4c^2xy+(x^2-y^2)^2 = 0$.
    Hyperbola equation $x^2-y^2 = a^2$: $4x^2y^2+a^2(x^2-y^2) = 0$.

51. $b^2x^2-a^2y^2 = a^2b^2$.

53. $(a^2+b^2)^2 = \dfrac{a^2}{l^2}-\dfrac{b^2}{m^2}$.

57. $4x^2y^2 = a^2y^2-b^2x^2$.

58. Hyperbola equation $xy = c^2$: $2x^2y^2 = c^2(x^2+y^2)$.
Hyperbola equation $x^2-y^2 = a^2$: $(x^2-y^2)^2 = a^2(x^2+y^2)$.

62. $17(x^2-y^2)+38xy = 0$.

63. $2x^2-3xy-2y^2-x+7y = 0$.

64. $2(x^2-y^2) = a^2-b^2$.

65. (1) As. $3x-2y+5 = 0$, $2x-y+3 = 0$.
C.H. $6x^2-7xy+2y^2+19x-11y+39 = 0$.
(2) As. $2x-y+8 = 0$, $2x-11y+6 = 0$.
C.H. $4x^2-24xy+11y^2+28x-94y+100 = 0$.

66. Perpendicular.

67. (1) $y\pm x = \pm 4$. (2) $x\pm\sqrt{2}y+2 = 0$.

82. $r^2\sin 2\theta = 2c^2$.

83. $r^2 = a^2\cos^2\theta-b^2\sin^2\theta$.

84. $\dfrac{l}{2r} = 1-e\cos\theta$, or $\dfrac{(2x-ae)^2}{a^2}-\dfrac{4y^2}{b^2} = 1$.

88. $l^2-2\sqrt{2}lr\cos\theta+r^2\left(1-\tan^2\dfrac{\alpha}{2}\right) = 0$.

EXAMPLES 15 (*p.* 333)

1. $-\dfrac{5+24\sqrt{2}}{49}$.

3. $11x+13y = 85$.

4. $y = nx$, where $n = \dfrac{2+\sqrt{2}+\sqrt{6}}{\sqrt{3}-2\sqrt{2}-1}$ or $-\dfrac{2+\sqrt{2}-\sqrt{6}}{\sqrt{3}+2\sqrt{2}+1}$.

5. $\dfrac{a}{a\sqrt{3}-2b}$.

8. $\frac{1}{2}hk\sin^3\omega$.

9. $nx-y = nh-k$.

11. $xy = kx+hy$.

12. $4xy\sin\omega = c$.

13. $\left(\dfrac{1}{c}, \dfrac{1}{c}\right)$.

14. $x(\cos\omega-c)+y(1-c\cos\omega) = 0$.

15. $2(x^2+y^2)\cos\omega+2xy(1+\cos^2\omega)-x(k+h\cos\omega)-$
$-y(h+k\cos\omega) = 0.$

16. $xy(h-k) = c(x-h)(x-k).$

19. $(12-2\sqrt{3}, 4-6\sqrt{3})$; $2(10-3\sqrt{3})^{\frac{1}{2}}$.

20. $x^2+xy+y^2-5x-y-74 = 0.$

22. $x+4y-18 = 0.$

23. $x^2+y^2+2xy\cos\omega-\alpha x-\beta y = 0.$

24. $x+2y+10 = 0.$

27. $x-\frac{1}{2}(t_1+t_2)y+\alpha t_1 t_2 = 0$; $x-ty+\alpha t^2 = 0.$

31. $(t\cos\omega+1)(y-2\alpha t)+(t+\cos\omega)(x-\alpha t^2) = 0.$

46. $x^2+y^2+2xy\cos\omega = 4c^2\cos\omega.$

47. (1) $2xy = kx+hy.$
   (2) $(x^2-2xy\cos\omega+y^2)(xy-c^2)-l^2xy = 0$, where $2l$ is the given length.

49. $2x^2+\alpha y = 0.$

EXAMPLES FROM H.C. PAPERS (*p.* 339)

1. (i) $x-3y = 25$; $13x-9y = 25$. (ii) $(-5, -10)$. (iii) 30.

2. (1) $4x+3y = 20$; $2x = y$; $x+1 = 0$.
   (2) 26° 34′; 36° 52′; 116° 34′.
   (3) 15.

4. $x = y.$

5. $\tan^{-1}\dfrac{m_1 l_2-m_2 l_1}{m_1 m_2+l_1 l_2}$. $3x+y = 8$; $x-3y = 6$.

6. O lies between P_1 and P_2 if the sign is negative.
   $21x-27y+3340 = 0.$

7. $bc+ca+ab.$

8. $(-1, 1)$, $(7, 7)$; $(-\frac{217}{25}, -\frac{119}{25})$, $(-\frac{17}{25}, \frac{31}{25})$.

9. $\frac{50}{23}$.

10. $\dfrac{m_1 x_2 \pm m_2 x_1}{m_1 \pm m_2}, \dfrac{m_1 y_2 \pm m_2 y_1}{m_1 \pm m_2}.$

11. $a+b = 0.$

12. $3x+2y = 24.$

13. $\dfrac{x}{a}+\dfrac{y}{b} = 1.$

14. $y = 2x$, $x+2y = 0$.

15. $2\sqrt{\{(g^2-ac)(1+m^2)\}}/a+hm$.

16. $k = 2$; 1.

19. 20·52.

22. $2g_2(g_1-g_2)+2f_2(f_1-f_2) = c_1-c_2$.

23. Inside the smaller circle.

26. $x^2+y^2-4ax+3a^2 = 0$; $x^2+y^2-4ax+4ay+7a^2 = 0$.

27. $x^2 = 2cy+l^2-c^2$.

28. $x^2(1+2gl+cl^2)+2xy(lf+mg+clm)+y^2(1+2fm+cm^2) = 0$.

$$-\frac{l(a-b)+2hm}{2(bl^2+am^2-2hlm)};\ \frac{m(a-b)-2hl}{2(bl^2+am^2-2hlm)}.$$

29. $h(x^2+y^2)-(h^2+pq)x-h(p-q)y = 0$.

30. $(x-a)\cos\theta+(y-b)\sin\theta = r$.

$$\left(\frac{r_1-r_2}{r_1+r_2}a, b\right);\ \left(\frac{r_1+r_2}{r_1-r_2}a, b\right).$$

31. Circles through the points where the given circle is cut by the line $lx+my+n = 0$.
$x^2+y^2+4x-2y-4 = 0$; $(-2, 1)$; 3.

34. $xx'+yy' = r^2$; $y^2 = ax+c$.

36. $6x+9y+1 = 0$.

37. $(-a, -b)$; $(1, \sqrt{3})$, $(1, -\sqrt{3})$.

38. $2\sqrt{(p^2+k\sin^2\alpha)}\operatorname{cosec}^2\alpha$.

39. (1) $4x-3y = 4$. (ii) $(1, 0)$. (iii) $4\frac{2}{5}$.

40. $\frac{4}{m^2}\sqrt{\{a(1+m^2)(a-mc)\}}$.

42. $x-\frac{1}{2}(t+m)y+atm = 0$.

44. 1 or $\frac{1}{3}$.

45. $t_1 = -\left(t+\frac{2}{t}\right)$.

46. $y^2 = 4a(x-8a)$.

47. $c+am(m^2+2) = 0$.

49. (i) $at^3+t(2a-h)-k = 0$.
(ii) $a^2t^4+2at^2(2a+g)+4fat+c = 0$.

50. $\frac{1}{8a}(y_1-y_2)(y_2-y_3)(y_3-y_1)$.

53. $y^2 = 2a(x+a)$.

54. $2a^2$.

56. $(2{\cdot}592,\ 76^\circ\ 22')\ (5{\cdot}393,\ 328^\circ\ 38')$.

57. $\dfrac{l}{r} = \dfrac{2\sin^3\dfrac{\alpha}{2}}{\sin\left(\dfrac{3\alpha}{2}-\theta\right)}$.

58. $\frac{10}{3}$; $\frac{4}{3}$; $(0, 5)$, $(0, 1)$.

59. $\dfrac{x^2}{16}+\dfrac{y^2}{25} = 1$; $x^2+y^2 = 25$; $x^2+y^2 = 41$. $(9, -14), (-\frac{8}{5}, \frac{89}{5})$.

60. $\frac{4}{5}$.

61. $x-4y+5 = 0$; $7x-20y-29 = 0$.

63. $x\sqrt{(d^2-b^2)}\pm y\sqrt{(a^2-c^2)}\pm\sqrt{(a^2d^2-b^2c^2)} = 0$.

64. $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2} = \dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}$.

65. $(a\cos\theta,\ b\sin\theta)$; $\tan^{-1}\dfrac{b}{a}$; $x+y = 0$.

67. $a^2+b^2 = a^4l^2+b^4m^2$.

68. $\dfrac{2\sqrt{2}ab}{\sqrt{(a^2+b^2)}}$.

69. $lm = -\dfrac{b^2}{a^2}$.

70. $a(1\pm e\cos\phi)$.

72. $(b^2x^2+a^2y^2-a^2b^2)(lX+mY+n)^2-$
$-2(b^2Xx+a^2Yy-a^2b^2)(lx+my+n)(lX+mY+n)+$
$+(b^2X^2+a^2Y^2-a^2b^2)(lx+my+n)^2 = 0$.

73. $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2} = 1$; $\left(-\dfrac{a^2l}{n}, -\dfrac{b^2m}{n}\right)$.

74. $\dfrac{x}{a}\cos\phi+\dfrac{y}{b}\sin\phi = 1$; $\dfrac{ax}{\cos\phi}-\dfrac{by}{\sin\phi} = a^2-b^2$.

77. $\dfrac{1}{m}, \dfrac{c^2l}{m}$.

82. $bx\pm ay = 0$.

85. $x+\alpha\beta y = c(\alpha+\beta)$.

86. $\dfrac{2at_1t_2}{t_1+t_2}, \dfrac{2a}{t_1+t_2}$.

88. Asymptotes: $2x+y=1$, $x-2y=1$. Centre: $(\frac{3}{5}, -\frac{1}{5})$.
Axes: $3x-y=2$, $x+3y=0$.

89. $\dfrac{a^2x}{x'}+\dfrac{b^2y}{y'}=a^2+b^2$.

$\dfrac{x^2}{a^2+b^2}-\dfrac{y^2}{b^2}=1$.

90. 45°, 135°.

91. $a^2p^2-b^2q^2=1$.

92. (i) $4c^2x^2-4c\lambda xy+\lambda^2y^2-\lambda^2cy=0$.
(ii) $cx^2-\lambda xy-cy^2+c^2y=0$.

95. $abcd=1$.

97. $(2,-1)$; 6 and 2; $x+y=1$, $x-y=3$. $2(y-x)=3$.

98. $mm'-(lm'+l'm)\cos\omega+ll'=0$. $\left(\dfrac{c}{t\cos\omega}, \dfrac{ct}{\cos\omega}\right)$.

99. $2\cos^2\beta+2e\cos\alpha\cos\beta-(1-e^2)=0$.

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